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THE
METHOD of FLUXIONS
AND
INFINITE SERIES;

WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, *K^t*.
Late President of the Royal Society.

*Translated from the AUTHOR's LATIN ORIGINAL
not yet made publick.*

To which is subjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Consisting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatise
A compleat Institution for the use of LEARNERS.

By *JOHN COLSON*, M. A. and F. R. S.
Master of Sir *Joseph Williamson's* free Mathematical-School at *Rocheſter*.

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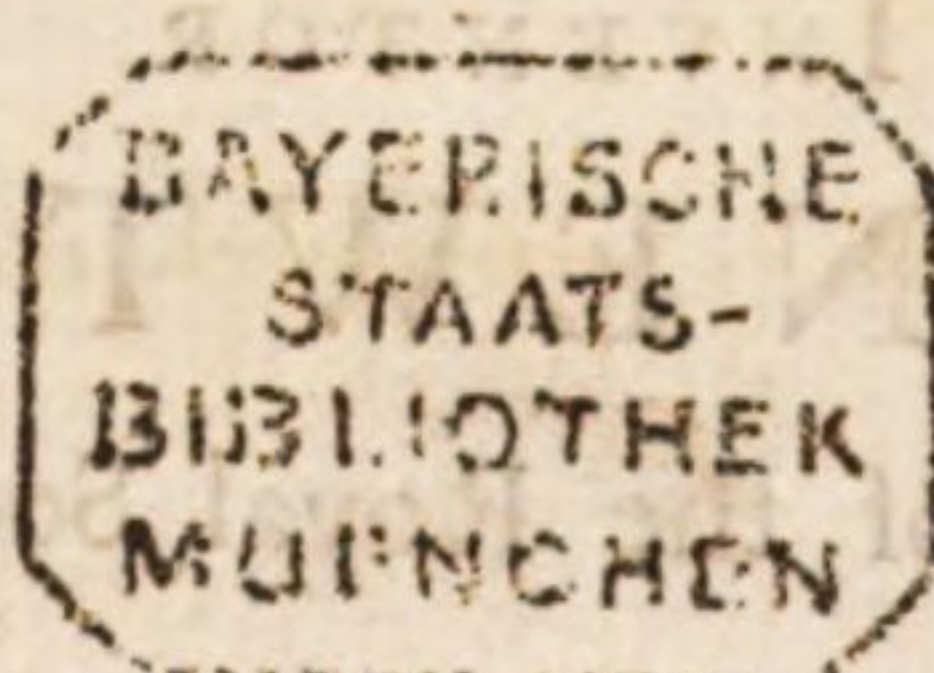
THE
METHOD OF FLUXIONS

AND

INFINITE SERIES;

WITH ITS

APPLICATION TO THE GEOMETRY OF CURVE-LINES.



Translated from the AUTHOR'S LATIN ORIGINAL
and with many additions.

To which is added,

A PRACTICAL COMMENT upon the whole Work,

of Consulting of

A NOTATION, ILLUSTRATIONS, and SUPPLEMENTS,

in order to make the Work

more useful and complete.

BY JOHN COLESON, M.A. &c.

Author of the "NEW METHOD OF FLUXIONS," &c.

MDCCLXXI.



T O

William Jones Esq; F.R.S.

S I R,

IT was a laudable custom among the ancient Geometers, and very worthy to be imitated by their Successors, to address their Mathematical labours, not so much to Men of eminent rank and station in the world, as to Persons of distinguish'd merit and proficiencie in the same Studies. For they knew very well, that such only could be competent Judges of their Works, and would receive them with the esteem they might deserve. So far at least I can copy after those great Originals, as to chuse a Patron for these Speculations, whose known skill and abilities in such matters will enable him to judge, and whose known candor will incline him to judge favourably, of the share I have had in the present performance. For as to the fundamental part of the Work, of which I am only the Interpreter, I know it cannot but please you; it will need no protection, nor can it receive a greater recommendation, than to bear the name of its illustrious Author. However, it very naturally applies itself to you, who had the honour (for I am sure you think it so) of the Author's friendship and familiarity in his life-time; who had his own consent to publish an elegant edition of some of his pieces, of a nature not very different from this; and who have so just an esteem for, as well as knowledge of, his other most sublime, most admirable, and justly celebrated Works.

But besides these motives of a publick nature, I had others that more nearly concern myself. The many personal obligations I have received from you, and your generous manner of conferring them, require all the testimonies of gratitude in my power. Among the rest, give me leave to mention one, (tho' it be a privilege I have enjoy'd in common with many others, who have the happiness of your acquaintance,) which is, the free access you have always allow'd me, to your copious Collection of whatever is choice and excellent in the Mathematicks. Your judgment and industry, in collecting those valuable *κειμήλια*, are not more conspicuous, than the freedom and readiness with which you communicate them, to all such who you know will apply them to their proper use, that is, to the general improvement of Science.

Before I take my leave, permit me, good Sir, to join my wishes to those of the publick, that your own useful Lucubrations may see the light, with all convenient speed; which, if I rightly conceive of them, will be an excellent methodical Introduction, not only to the mathematical Sciences in general, but also to these, as well as to the other curious and abstruse Speculations of our great Author. You are very well apprized, as all other good Judges must be, that to illustrate him is to cultivate real Science, and to make his Discoveries easy and familiar, will be no small improvement in Mathematicks and Philosophy.

That you will receive this address with your usual candor, and with that favour and friendship I have so long and often experienced, is the earnest request of,

S I R,

Your most obedient humble Servant,

J. COLSON.



T H E
P R E F A C E.

I Cannot but very much congratulate with my Mathematical Readers, and think it one of the most fortunate circumstances of my Life, that I have it in my power to present the publick with a most valuable Anecdote, of the greatest Master in Mathematical and Philosophical Knowledge, that ever appear'd in the World. And so much the more, because this Anecdote is of an elementary nature, preparatory and introductory to his other most arduous and sublime Speculations, and intended by himself for the instruction of Novices and Learners. I therefore gladly embraced the opportunity that was put into my hands, of publishing this posthumous Work, because I found it had been composed with that view and design. And that my own Country-men might first enjoy the benefit of this publication, I resolved upon giving it in an *English* Translation, with some additional Remarks of my own. I thought it highly injurious to the memory and reputation of the great Author, as well as invidious to the glory of our own Nation, that so curious and useful a piece should be any longer suppress'd, and confined to a few private hands, which ought to be communicated to all the learned World for general Instruction. And more especially at a time when the Principles of the Method here taught have been scrupulously sifted and examin'd, have been vigorously opposed and (we may say) ignominiously rejected as insufficient, by some Mathematical Gentlemen, who seem not to have derived their knowledge of them from their only true Source, that is, from our Author's own Treatise wrote expressly to explain them. And on the other hand, the Principles of this Method have been zealously and commendably defended by other Mathematical Gentlemen, who yet

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seem to have been as little acquainted with this Work, (or at least to have over-look'd it,) the only genuine and original Fountain of this kind of knowledge. For what has been elsewhere deliver'd by our Author, concerning this Method, was only accidental and occasional, and far from that copiousness with which he treats of it here, and illustrates it with a great variety of choice Examples.

The learned and ingenious Dr. *Pemberton*, as he acquaints us in his View of Sir *Isaac Newton's* Philosophy, had once a design of publishing this Work, with the consent and under the inspection of the Author himself; which if he had then accomplish'd, he would certainly have deserved and received the thanks of all lovers of Science. The Work would have then appear'd with a double advantage, as receiving the last Emendations of its great Author, and likewise in passing through the hands of so able an Editor. And among the other good effects of this publication, possibly it might have prevented all or a great part of those Disputes, which have since been rais'd, and which have been so strenuously and warmly pursued on both sides, concerning the validity of the Principles of this Method. They would doubtless have been placed in so good a light, as would have cleared them from any imputation of being in any wise defective, or not sufficiently demonstrated. But since the Author's Death, as the Doctor informs us, prevented the execution of that design, and since he has not thought fit to resume it hitherto, it became needful that this publication should be undertook by another, tho' a much inferior hand.

For it was now become highly necessary, that at last the great Sir *Isaac* himself should interpose, should produce his genuine Method of Fluxions, and bring it to the test of all impartial and considerate Mathematicians; to shew its evidence and simplicity, to maintain and defend it in his own way, to convince his Opponents, and to teach his Disciples and Followers upon what grounds they should proceed in vindication of the Truth and Himself. And that this might be done the more easily and readily, I resolv'd to accompany it with an ample Commentary, according to the best of my skill, and (I believe) according to the mind and intention of the Author, wherever I thought it needful; and particularly with an Eye to the fore-mention'd Controversy. In which I have endeavour'd to obviate the difficulties that have been rais'd, and to explain every thing in so full a manner, as to remove all the objections of any force, that have been any where made, at least such as have occur'd to my observation. If what is here advanced, as there is good reason

son to hope, shall prove to the satisfaction of those Gentlemen, who first started these objections, and who (I am willing to suppose) had only the cause of Truth at heart; I shall be very glad to have contributed any thing, towards the removing of their Scruples. But if it shall happen otherwise, and what is here offer'd should not appear to be sufficient evidence, conviction, and demonstration to them; yet I am persuaded it will be such to most other thinking Readers, who shall apply themselves to it with unprejudiced and impartial minds; and then I shall not think my labour ill bestow'd. It should however be well consider'd by those Gentlemen, that the great number of Examples they will find here, to which the Method of Fluxions is successfully apply'd, are so many vouchers for the truth of the Principles, on which that Method is founded. For the Deductions are always conformable to what has been derived from other uncontroverted Principles, and therefore must be acknowledg'd as true. This argument should have its due weight, even with such as cannot, as well as with such as will not, enter into the proof of the Principles themselves. And the *hypothesis* that has been advanced to evade this conclusion, of one error in reasoning being still corrected by another equal and contrary to it, and that so regularly, constantly, and frequently, as it must be suppos'd to do here; this *hypothesis*, I say, ought not to be seriously refuted, because I can hardly think it is seriously proposed.

The chief Principle, upon which the Method of Fluxions is here built, is this very simple one, taken from the Rational Mechanicks; which is, That Mathematical Quantity, particularly Extension, may be conceived as generated by continued local Motion; and that all Quantities whatever, at least by analogy and accommodation, may be conceived as generated after a like manner. Consequently there must be comparative Velocities of increase and decrease, during such generations, whose Relations are fixt and determinable, and may therefore (problematically) be proposed to be found. This Problem our Author here solves by the help of another Principle, not less evident; which supposes that Quantity is infinitely divisible, or that it may (mentally at least) so far continually diminish, as at last, before it is totally extinguish'd, to arrive at Quantities that may be call'd vanishing Quantities, or which are infinitely little, and less than any assignable Quantity. Or it supposes that we may form a Notion, not indeed of absolute, but of relative and comparative infinity. 'Tis a very just exception to the Method of Indivisibles, as also to the foreign infinitesimal Method, that they have recourse at once to

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infinitely little Quantities, and infinite orders and gradations of these, not relatively but absolutely such. They assume these Quantities *simul & semel*, without any ceremony, as Quantities that actually and obviously exist, and make Computations with them accordingly; the result of which must needs be as precarious, as the absolute existence of the Quantities they assume. And some late Geometricians have carry'd these Speculations, about real and absolute Infinity, still much farther, and have rais'd imaginary Systems of infinitely great and infinitely little Quantities, and their several orders and properties; which, to all sober Inquirers into mathematical Truths, must certainly appear very notional and visionary.

These will be the inconveniencies that will arise, if we do not rightly distinguish between absolute and relative Infinity. Absolute Infinity, as such, can hardly be the object either of our Conceptions or Calculations, but relative Infinity may, under a proper regulation. Our Author observes this distinction very strictly, and introduces none but infinitely little Quantities that are relatively so; which he arrives at by beginning with finite Quantities, and proceeding by a gradual and necessary progress of diminution. His Computations always commence by finite and intelligible Quantities; and then at last he inquires what will be the result in certain circumstances, when such or such Quantities are diminish'd *in infinitum*. This is a constant practice even in common Algebra and Geometry, and is no more than descending from a general Proposition, to a particular Case which is certainly included in it. And from these easy Principles, managed with a vast deal of skill and sagacity, he deduces his Method of Fluxions; which if we consider only so far as he himself has carry'd it, together with the application he has made of it, either here or elsewhere, directly or indirectly, expressly or tacitely, to the most curious Discoveries in Art and Nature, and to the sublimest Theories: We may deservedly esteem it as the greatest Work of Genius, and as the noblest Effort that ever was made by the Human Mind. Indeed it must be own'd, that many useful Improvements, and new Applications, have been since made by others, and probably will be still made every day. For it is no mean excellence of this Method, that it is doubtless still capable of a greater degree of perfection; and will always afford an inexhaustible fund of curious matter, to reward the pains of the ingenious and industrious Analyst.

As I am desirous to make this as satisfactory as possible, especially to the very learned and ingenious Author of the Discourse call'd *The Analyst*, whose eminent Talents I acknowledge myself to have a

great veneration for ; I shall here endeavour to obviate some of his principal Objections to the Method of Fluxions, particularly such as I have not touch'd upon in my Comment, which is soon to follow.

He thinks our Author has not proceeded in a demonstrative and scientific matter, in his *Princip. lib. 2. lem. 2.* where he deduces the Moment of a Rectangle, whose Sides are supposed to be variable Lines. I shall represent the matter Analytically thus, agreeably (I think) to the mind of the Author.

Let X and Y be two variable Lines, or Quantities, which at different periods of time acquire different values, by flowing or increasing continually, either equably or alike inequably. For instance, let there be three periods of time, at which X becomes $A - \frac{1}{2}a$, A, and $A + \frac{1}{2}a$; and Y becomes $B - \frac{1}{2}b$, B, and $B + \frac{1}{2}b$ successively and respectively; where A, a, B, b, are any quantities that may be assumed at pleasure. Then at the same periods of time the variable Product or Rectangle XY will become $A - \frac{1}{2}a \times B - \frac{1}{2}b$, AB, and $A + \frac{1}{2}a \times B + \frac{1}{2}b$, that is, $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$, AB, and $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Now in the interval from the first period of time to the second, in which X from being $A - \frac{1}{2}a$ is become A, and in which Y from being $B - \frac{1}{2}b$ is become B, the Product XY from being $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$ becomes AB; that is, by Subtraction, its whole Increment during that interval is $\frac{1}{2}aB + \frac{1}{2}bA - \frac{1}{4}ab$. And in the interval from the second period of time to the third, in which X from being A becomes $A + \frac{1}{2}a$, and in which Y from being B becomes $B + \frac{1}{2}b$, the Product XY from being AB becomes $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$; that is, by Subtraction, its whole Increment during that interval is $\frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. Add these two Increments together, and we shall have $aB + bA$ for the compleat Increment of the Product XY, during the whole interval of time, while X flow'd from the value $A - \frac{1}{2}a$ to $A + \frac{1}{2}a$, or Y flow'd from the value $B - \frac{1}{2}b$ to $B + \frac{1}{2}b$. Or it might have been found by one Operation, thus: While X flows from $A - \frac{1}{2}a$ to A, and thence to $A + \frac{1}{2}a$, or Y flows from $B - \frac{1}{2}b$ to B, and thence to $B + \frac{1}{2}b$, the Product XY will flow from $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$ to AB, and thence to $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$; therefore by Subtraction the whole Increment during that interval of time will be $aB + bA$. Q. E. D.

This may easily be illustrated by Numbers thus: Make A, a, B, b, equal to 9, 4, 15, 6, respectively; (or any other Numbers to be assumed at pleasure.) Then the three successive values of X will be 7, 9, 11, and the three successive values of Y will be 12, 15, 18, respectively.

respectively. Also the three successive values of the Product XY will be 84, 135, 198. But $aB + bA = 4 \times 15 + 6 \times 9 = 114 = 198 - 84$. Q. E. O.

Thus the Lemma will be true of any conceivable finite Increments whatever; and therefore by way of Corollary, it will be true of infinitely little Increments, which are call'd Moments, and which was the thing the Author principally intended here to demonstrate. But in the case of Moments it is to be consider'd, that X , or definitely $A - \frac{1}{2}a$, A , and $A + \frac{1}{2}a$, are to be taken indifferently for the same Quantity; as also Y , and definitely $B - \frac{1}{2}b$, B , $B + \frac{1}{2}b$. And the want of this Consideration has occasion'd not a few perplexities.

Now from hence the rest of our Author's Conclusions, in the same Lemma, may be thus derived something more explicitly. The Moment of the Rectangle AB being found to be $Ab + aB$, when the contemporary Moments of A and B are represented by a and b respectively; make $B = A$, and therefore $b = a$, and then the Moment of $A \times A$, or A^2 , will be $Aa + aA$, or $2aA$. Again, make $B = A^2$, and therefore $b = 2aA$, and then the Moment of $A \times A^2$, or A^3 , will be $2aA^2 + aA^2$, or $3aA^2$. Again, make $B = A^3$, and therefore $b = 3aA^2$, and then the Moment of $A \times A^3$, or A^4 , will be $3aA^3 + aA^3$, or $4aA^3$. Again, make $B = A^4$, and therefore $b = 4aA^3$, and then the Moment of $A \times A^4$, or A^5 , will be $4aA^4 + aA^4$, or $5aA^4$. And so on *in infinitum*. Therefore in general, assuming m to represent any integer affirmative Number, the Moment of A^m will be maA^{m-1} .

Now because $A^m \times A^{-m} = 1$, (where m is any integer affirmative Number,) and because the Moment of Unity, or any other constant quantity, is $= 0$; we shall have $A^m \times \text{Mom. } A^{-m} + A^{-m} \times \text{Mom. } A^m = 0$, or $\text{Mom. } A^{-m} = -A^{-2m} \times \text{Mom. } A^m$. But $\text{Mom. } A^m = maA^{m-1}$, as found before; therefore $\text{Mom. } A^{-m} = -A^{-2m} \times maA^{m-1} = -maA^{-m-1}$. Therefore the Moment of A^m will be maA^{m-1} , when m is any integer Number, whether affirmative or negative.

And universally, if we put $A^{\frac{m}{n}} = B$, or $A^m = B^n$, where m and n may be any integer Numbers, affirmative or negative; then we shall have $maA^{m-1} = nbB^{n-1}$, or $b = \frac{maA^{m-1}}{nA^{\frac{mn-n}{n}}} = \frac{m}{n}aA^{\frac{m}{n}-1}$, which

is the Moment of B , or of $A^{\frac{m}{n}}$. So that the Moment of A^m will be

be still maA^{m-1} , whether m be affirmative or negative, integer or fraction.

The Moment of AB being $bA + aB$, and the Moment of CD being $dC + cD$; suppose $D = AB$, and therefore $d = bA + aB$, and then by Substitution the Moment of ABC will be $bA + aB \times C + cAB = bAC + aBC + cAB$. And likewise the Moment of $A^m B^n$ will be $nbB^{n-1}A^m + maA^{m-1}B^n$. And so of any others.

Now there is so near a connexion between the Method of Moments and the Method of Fluxions, that it will be very easy to pass from the one to the other. For the Fluxions or Velocities of increase, are always proportional to the contemporary Moments. Thus if for $A, B, C, \&c.$ we write $x, y, z, \&c.$ for $a, b, c, \&c.$ we may write $\dot{x}, \dot{y}, \dot{z}, \&c.$ Then the Fluxion of xy will be $\dot{x}y + x\dot{y}$, the Fluxion of x^m will be $m\dot{x}x^{m-1}$, whether m be integer or fraction, affirmative or negative; the Fluxion of xyz will be $\dot{x}yz + x\dot{y}z + xy\dot{z}$, and the Fluxion of $x^m y^n$ will be $m\dot{x}x^{m-1}y^n + nx^m\dot{y}y^{n-1}$. And so of the rest.

Or the former Inquiry may be placed in another view, thus: Let A and $A + a$ be two successive values of the variable Quantity X , as also B and $B + b$ be two successive and contemporary values of Y ; then will AB and $AB + aB + bA + ab$ be two successive and contemporary values of the variable Product XY . And while X , by increasing perpetually, flows from its value A to $A + a$, or Y flows from B to $B + b$; XY at the same time will flow from AB to $AB + aB + bA + ab$, during which time its whole Increment, as appears by Subtraction, will become $aB + bA + ab$. Or in Numbers thus: Let A, a, B, b , be equal to 7, 4, 12, 6, respectively; then will the two successive values of X be 7, 11, and the two successive values of Y will be 12, 18. Also the two successive values of the Product XY will be 84, 198. But the Increment $aB + bA + ab = 48 + 42 + 24 = 114 = 198 - 84$, as before.

And thus it will be as to all finite Increments: But when the Increments become Moments, that is, when a and b are so far diminish'd, as to become infinitely less than A and B ; at the same time ab will become infinitely less than either aB or bA , (for $aB. ab :: B. b$, and $bA. ab :: A. a$.) and therefore it will vanish in respect of them. In which case the Moment of the Product or Rectangle will be $aB + bA$, as before. This perhaps is the more obvious and direct way of proceeding, in the present Inquiry; but, as there was room for choice, our Author thought fit to chuse the former way,

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as the more elegant, and in which he was under no necessity of having recourse to that Principle, that quantities arising in an Equation, which are infinitely less than the others, may be neglected or expunged in comparison of those others. Now to avoid the use of this Principle, tho' otherwise a true one, was all the Artifice used on this occasion, which certainly was a very fair and justifiable one.

I shall conclude my Observations with considering and obviating the Objections that have been made, to the usual Method of finding the Increment, Moment, or Fluxion of any indefinite power x^n of the variable quantity x , by giving that Investigation in such a manner, as to leave (I think) no room for any just exceptions to it. And the rather because this is a leading point, and has been strangely perverted and misrepresented.

In order to find the Increment of the variable quantity or power x^n , (or rather its relation to the Increment of x , consider'd as given; because Increments and Moments can be known only by comparison with other Increments and Moments, as also Fluxions by comparison with other Fluxions;) let us make $x^n = y$, and let X and Y be any synchronous Augments of x and y . Then by the hypothesis we shall have the Equation $\overline{x + X}^n = y + Y$; for in any Equation the variable Quantities may always be increased by their synchronous Augments, and yet the Equation will still hold good. Then by our Author's famous Binomial Theorem we shall have $y + Y = x^n + nx^{n-1}X + n \times \frac{n-1}{2} x^{n-2}X^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}X^3$, &c. or removing the equal Quantities y and x^n , it will be $Y = nx^{n-1}X + n \times \frac{n-1}{2} x^{n-2}X^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} x^{n-3}X^3$, &c. So that when X denotes the given Increment of the variable quantity x , Y will here denote the synchronous Increment of the indefinite power y or x^n ; whose value therefore, in all cases, may be had from this Series. Now that we may be sure we proceed regularly, we will verify this thus far, by a particular and familiar instance or two. Suppose $n = 2$, then $Y = 2xX + X^2$. That is, while x flows or increases to $x + X$, x^2 in the same time, by its Increment $Y = 2xX + X^2$, will increase to $x^2 + 2xX + X^2$, which we otherwise know to be true. Again, suppose $n = 3$, then $Y = 3x^2X + 3xX^2 + X^3$. Or while x increases to $x + X$, x^3 by its Increment $Y = 3x^2X + 3xX^2 + X^3$ will increase to $x^3 + 3x^2X + 3xX^2 + X^3$. And so in all other particular cases, whereby we may plainly perceive, that this general Conclusion must be certain and indubitable.

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This Series therefore will be always true, let the Augments X and Y be ever so great, or ever so little; for the truth does not at all depend on the circumstance of their magnitude. Nay, when they are infinitely little, or when they become Moments, it must be true also, by virtue of the general Conclusion. But when X and Y are diminish'd *in infinitum*, so as to become at last infinitely little, the greater powers of X must needs vanish first, as being relatively of an infinitely less value than the smaller powers. So that when they are all expunged, we shall necessarily obtain the Equation $Y = nx^{n-1}X$; where the remaining Terms are likewise infinitely little, and consequently would vanish, if there were other Terms in the Equation, which were (relatively) infinitely greater than themselves. But as there are not, we may securely retain this Equation, as having an undoubted right so to do; and especially as it gives us an useful piece of information, that X and Y , tho' themselves infinitely little, or vanishing quantities, yet they vanish in proportion to each other as 1 to nx^{n-1} . We have therefore learn'd at last, that the Moment by which x increases, or X , is to the contemporary Moment by which x^n increases, or Y , as 1 is to nx^{n-1} . And their Fluxions, or Velocities of increase, being in the same proportion as their synchronous Moments, we shall have $nx^{n-1}\dot{x}$ for the Fluxion of x^n , when the Fluxion of x is denoted by $\dot{x}$.

I cannot conceive there can be any pretence to insinuate here, that any unfair artifices, any leger-de-main tricks, or any shifting of the hypothesis, that have been so severely complain'd of, are at all made use of in this Investigation. We have legitimately derived this general Conclusion in finite Quantities, that in all cases the relation of the Increments will be $Y = nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. of which one particular case is, when X and Y are supposed continually to decrease, till they finally terminate in nothing. But by thus continually decreasing, they approach nearer and nearer to the Ratio of 1 to nx^{n-1} , which they attain to at the very instant of their vanishing, and not before. This therefore is their ultimate Ratio, the Ratio of their Moments, Fluxions, or Velocities, by which x and x^n continually increase or decrease. Now to argue from a general Theorem to a particular case contain'd under it, is certainly one of the most legitimate and logical, as well as one of the most usual and useful ways of arguing, in the whole compass of the Mathematics. To object here, that after we have made X and Y to stand for some quantity, we are not at liberty to make them nothing, or no quantity, or vanishing quantities, is not an Objection against the

Method of Fluxions, but against the common Analyticks. This Method only adopts this way of arguing, as a constant practice in the vulgar Algebra, and refers us thither for the proof of it. If we have an Equation any how compos'd of the general Numbers $a, b, c,$ &c. it has always been taught, that we may interpret these by any particular Numbers at pleasure, or even by 0, provided that the Equation, or the Conditions of the Question, do not expressly require the contrary. For general Numbers, as such, may stand for any definite Numbers in the whole Numerical Scale; which Scale (I think) may be thus commodiously represented, &c. — 3, — 2, — 1, 0, 1, 2, 3, 4, &c. where all possible fractional Numbers, intermediate to these here express'd, are to be conceived as interpolated. But in this Scale the Term 0 is as much a Term or Number as any other, and has its analogous properties in common with the rest. We are likewise told, that we may not give such values to general Symbols afterwards, as they could not receive at first; which if admitted is, I think, nothing to the present purpose. It is always most easy and natural, as well as most regular, instructive, and elegant, to make our Inquiries as much in general Terms as may be, and to descend to particular cases by degrees, when the Problem is nearly brought to a conclusion. But this is a point of convenience only, and not a point of necessity. Thus in the present case, instead of descending from finite Increments to infinitely little Moments, or vanishing Quantities, we might begin our Computation with those Moments themselves, and yet we should arrive at the same Conclusions. As a proof of which we may consult our Author's own Demonstration of his Method, in pag. 24. of this Treatise. In short, to require this is just the same thing as to insist, that a Problem, which naturally belongs to Algebra, should be solved by common Arithmetick; which tho' possible to be done, by pursuing backwards all the steps of the general process, yet would be very troublesome and operose, and not so instructive, or according to the true Rules of Art.

But I am apt to suspect, that all our doubts and scruples about Mathematical Inferences and Argumentations, especially when we are satisfied that they have been justly and legitimately conducted, may be ultimately resolved into a species of infidelity and distrust. Not in respect of any implicate faith we ought to repose on meer human authority, tho' ever so great, (for that, in Mathematicks, we should utterly disclaim,) but in respect of the Science itself. We are hardly brought to believe, that the Science is so perfectly regular and uni-
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form, so infinitely consistent, constant, and accurate, as we shall really find it to be, when after long experience and reflexion we shall have overcome this prejudice, and shall learn to pursue it rightly. We do not readily admit, or easily comprehend, that Quantities have an infinite number of curious and subtile properties, some near and obvious, others remote and abstruse, which are all link'd together by a necessary connexion, or by a perpetual chain, and are then only discoverable when regularly and closely pursued; and require our trust and confidence in the Science, as well as our industry, application, and obstinate perseverance, our sagacity and penetration, in order to their being brought into full light. That Nature is ever consistent with herself, and never proceeds in these Speculations *per saltum*, or at random, but is infinitely scrupulous and solicitous, as we may say, in adhering to Rule and Analogy. That whenever we make any regular Positions, and pursue them through ever so great a variety of Operations, according to the strict Rules of Art; we shall always proceed through a series of regular and well-connected transmutations, (if we would but attend to 'em,) till at last we arrive at regular and just Conclusions. That no properties of Quantity are intirely destructible, or are totally lost and abolish'd, even tho' prosecuted to infinity itself; for if we suppose some Quantities to become infinitely great, or infinitely little, or nothing, or less than nothing, yet other Quantities that have a certain relation to them will only undergo proportional, and often finite alterations, will sympathize with them, and conform to 'em in all their changes; and will always preserve their analogical nature, form, or magnitude, which will be faithfully exhibited and discover'd by the result. This we may collect from a great variety of Mathematical Speculations, and more particularly when we adapt Geometry to Analyticks, and Curve-lines to Algebraical Equations. That when we pursue general Inquiries, Nature is infinitely prolifick in particulars that will result from them, whether in a direct subordination, or whether they branch out collaterally; or even in particular Problems, we may often perceive that these are only certain cases of something more general, and may afford good hints and assistances to a sagacious Analyst, for ascending gradually to higher and higher Disquisitions, which may be prosecuted more universally than was at first expected or intended. These are some of those Mathematical Principles, of a higher order, which we find a difficulty to admit, and which we shall never be fully convinced of, or know the whole use of, but from much practice and attentive consideration; but more especially by a diligent

perusal, and close examination, of this and the other Works of our illustrious Author. He abounded in these sublime views and inquiries, had acquired an accurate and habitual knowledge of all these, and of many more general Laws, or Mathematical Principles of a superior kind, which may not improperly be call'd *The Philosophy of Quantity*; and which, assisted by his great Genius and Sagacity, together with his great natural application, enabled him to become so compleat a Master in the higher Geometry, and particularly in the Art of Invention. This Art, which he possess'd in the greatest perfection imaginable, is indeed the sublimest, as well as the most difficult of all Arts, if it properly may be call'd such; as not being reducible to any certain Rules, nor can be deliver'd by any Precepts, but is wholly owing to a happy sagacity, or rather to a kind of divine Enthusiasm. To improve Inventions already made, to carry them on, when begun, to farther perfection, is certainly a very useful and excellent Talent; but however is far inferior to the Art of Discovery, as having a $\pi\epsilon\rho\omega$, or certain *data* to proceed upon, and where just method, close reasoning, strict attention, and the Rules of Analogy, may do very much. But to strike out new lights, to adventure where no footsteps had ever been set before, *nullius antè trita solo*; this is the noblest Endowment that a human Mind is capable of, is reserved for the chosen few *quos Jupiter æquus amavit*, and was the peculiar and distinguishing Character of our great Mathematical Philosopher. He had acquired a compleat knowledge of the Philosophy of Quantity, or of its most essential and most general Laws; had consider'd it in all views, had pursued it through all its disguises, and had traced it through all its Labyrinths and Recesses; in a word, it may be said of him not improperly, that he tortured and tormented Quantities all possible ways, to make them confess their Secrets, and discover their Properties.

The Method of Fluxions, as it is here deliver'd in this Treatise, is a very pregnant and remarkable instance of all these particulars. To take a cursory view of which, we may conveniently enough divide it into these three parts. The first will be the Introduction, or the Method of infinite Series. The second is the Method of Fluxions, properly so call'd. The third is the application of both these Methods to some very general and curious Speculations, chiefly in the Geometry of Curve-lines.

As to the first, which is the Method of infinite Series, in this the Author opens a new kind of Arithmetick, (new at least at the time of his writing this,) or rather he vastly improves the old. For

he

he extends the received Notation, making it compleatly universal, and shews, that as our common Arithmetick of Integers received a great Improvement by the introduction of decimal Fractions ; so the common Algebra or Analyticks, as an universal Arithmetick, will receive a like Improvement by the admission of his Doctrine of infinite Series, by which the same analogy will be still carry'd on, and farther advanced towards perfection. Then he shews how all complicate Algebraical Expressions may be reduced to such Series, as will continually converge to the true values of those complex quantities, or their Roots, and may therefore be used in their stead : whether those quantities are Fractions having multinomial Denominators, which are therefore to be resolved into simple Terms by a perpetual Division ; or whether they are Roots of pure Powers, or of affected Equations, which are therefore to be resolved by a perpetual Extraction. And by the way, he teaches us a very general and commodious Method for extracting the Roots of affected Equations in Numbers. And this is chiefly the substance of his Method of infinite Series.

The Method of Fluxions comes next to be deliver'd, which indeed is principally intended, and to which the other is only preparatory and subservient. Here the Author displays his whole skill, and shews the great extent of his Genius. The chief difficulties of this he reduces to the Solution of two Problems, belonging to the abstract or Rational Mechanicks. For the direct Method of Fluxions, as it is now call'd, amounts to this Mechanical Problem, *The length of the Space described being continually given, to find the Velocity of the Motion at any time proposed.* Also the inverse Method of Fluxions has, for a foundation, the Reverse of this Problem, which is, *The Velocity of the Motion being continually given, to find the Space described at any time proposed.* So that upon the compleat Analytical or Geometrical Solution of these two Problems, in all their varieties, he builds his whole Method.

His first Problem, which is, *The relation of the flowing Quantities being given, to determine the relation of their Fluxions*, he dispatches very generally. He does not propose this, as is usually done, *A flowing Quantity being given, to find its Fluxion* ; for this gives us too lax and vague an Idea of the thing, and does not sufficiently shew that Comparison, which is here always to be understood. Fluents and Fluxions are things of a relative nature, and suppose two at least, whose relation or relations should always be express'd by Equations. He requires therefore that all should be reduced to Equations, by which the relation of the flowing Quantities will be exhibited, and their comparative

comparative magnitudes will be more easily estimated; as also the comparative magnitudes of their Fluxions. And besides, by this means he has an opportunity of resolving the Problem much more generally than is commonly done. For in the usual way of taking Fluxions, we are confined to the Indices of the Powers, which are to be made Coefficients; whereas the Problem in its full extent will allow us to take any Arithmetical Progressions whatever. By this means we may have an infinite variety of Solutions, which tho' different in form, will yet all agree in the main; and we may always chuse the simplest, or that which will best serve the present purpose. He shews also how the given Equation may comprehend several variable Quantities, and by that means the Fluxional Equation may be found, notwithstanding any surd quantities that may occur, or even any other quantities that are irreducible, or Geometrically irrational. And all this is derived and demonstrated from the properties of Moments. He does not here proceed to second, or higher Orders of Fluxions, for a reason which will be assign'd in another place.

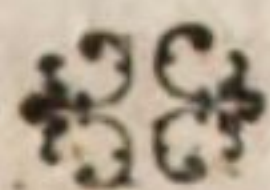
His next Problem is, *An Equation being proposed exhibiting the relation of the Fluxions of Quantities, to find the relation of those Quantities, or Fluents, to one another*; which is the direct Converse of the foregoing Problem. This indeed is an operose and difficult Problem, taking it in its full extent, and requires all our Author's skill and address; which yet he solves very generally, chiefly by the assistance of his Method of infinite Series. He first teaches how we may return from the Fluxional Equation given, to its corresponding finite Fluent or Algebraical Equation, when that can be done. But when it cannot be done, or when there is no such finite Algebraical Equation, as is most commonly the case, yet however he finds the Root of that Equation by an infinite converging Series, which answers the same purpose. And often he shews how to find the Root, or Fluent required, by an infinite number of such Series. His processes for extracting these Roots are peculiar to himself, and always contrived with much subtilty and ingenuity.

The rest of his Problems are an application or an exemplification of the foregoing. As when he determines the *Maxima* and *Minima* of quantities in all cases. When he shews the Method of drawing Tangents to Curves, whether Geometrical or Mechanical; or however the nature of the Curve may be defined, or refer'd to right Lines or other Curves. Then he shews how to find the Center or Radius of Curvature, of any Curve whatever, and that in a simple but general manner; which he illustrates by many curious Examples, and

and pursues many other ingenious Problems, that offer themselves by the way. After which he discusses another very subtle and intirely new Problem about Curves, which is, to determine the quality of the Curvity of any Curve, or how its Curvature varies in its progress through the different parts, in respect of equability or inequability.

He then applies himself to consider the Areas of Curves, and shews us how we may find as many Quadrable Curves as we please, or such whose Areas may be compared with those of right-lined Figures. Then he teaches us to find as many Curves as we please, whose Areas may be compared with that of the Circle, or of the Hyperbola, or of any other Curve that shall be assign'd; which he extends to Mechanical as well as Geometrical Curves. He then determines the Area in general of any Curve that may be proposed, chiefly by the help of infinite Series; and gives many useful Rules for ascertaining the Limits of such Areas. And by the way he squares the Circle and Hyperbola, and applies the Quadrature of this to the constructing of a Canon of Logarithms. But chiefly he collects very general and useful Tables of Quadratures, for readily finding the Areas of Curves, or for comparing them with the Areas of the Conic Sections; which Tables are the same as those he has publish'd himself, in his Treatise of Quadratures. The use and application of these he shews in an ample manner, and derives from them many curious Geometrical Constructions, with their Demonstrations.

Lastly, he applies himself to the Rectification of Curves, and shews us how we may find as many Curves as we please, whose Curve-lines are capable of Rectification; or whose Curve-lines, as to length, may be compared with the Curve-lines of any Curves that shall be assign'd. And concludes in general, with rectifying any Curve-lines that may be proposed, either by the assistance of his Tables of Quadratures, when that can be done, or however by infinite Series. And this is chiefly the substance of the present Work. As to the account that perhaps may be expected, of what I have added in my Annotations; I shall refer the inquisitive Reader to the Preface, which will go before that part of the Work.



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THE METHOD of FLUXIONS, AND INFINITE SERIES.

INTRODUCTION: *Or, the Resolution of Equations
by Infinite Series.*

I. **H**AVING observed that most of our modern Geometricians, neglecting the Synthetical Method of the Ancients, have apply'd themselves chiefly to the cultivating of the Analytical Art; by the assistance of which they have been able to overcome so many and so great difficulties, that they seem to have exhausted all the Speculations of Geometry, excepting the Quadrature of Curves, and some other matters of a like nature, not yet intirely discuss'd: I thought it not amiss, for the sake of young Students in this Science, to compose the following Treatise, in which I have endeavour'd to enlarge the Boundaries of Analyticks, and to improve the Doctrine of Curve-lines.

2. Since there is a great conformity between the Operations in Species, and the same Operations in common Numbers; nor do they seem to differ, except in the Characters by which they are re-
B presented,

presented, the first being general and indefinite, and the other definite and particular: I cannot but wonder that no body has thought of accommodating the lately-discover'd Doctrine of Decimal Fractions in like manner to Species, (unless you will except the Quadrature of the Hyberbola by Mr. *Nicolas Mercator*;) especially since it might have open'd a way to more abstruse Discoveries. But since this Doctrine of Species, has the same relation to Algebra, as the Doctrine of Decimal Numbers has to common Arithmetick; the Operations of Addition, Subtraction, Multiplication, Division, and Extraction of Roots, may easily be learned from thence, if the Learner be but skill'd in Decimal Arithmetick, and the Vulgar Algebra, and observes the correspondence that obtains between Decimal Fractions and Algebraick Terms infinitely continued. For as in Numbers, the Places towards the right-hand continually decrease in a Decimal or Subdecuple Proportion; so it is in Species respectively, when the Terms are disposed, (as is often enjoin'd in what follows,) in an uniform Progression infinitely continued, according to the Order of the Dimensions of any Numerator or Denominator. And as the convenience of Decimals is this, that all vulgar Fractions and Radicals, being reduced to them, in some measure acquire the nature of Integers, and may be managed as such; so it is a convenience attending infinite Series in Species, that all kinds of complicate Terms, (such as Fractions whose Denominators are compound Quantities, the Roots of compound Quantities, or of affected Equations, and the like,) may be reduced to the Class of simple Quantities; that is, to an infinite Series of Fractions, whose Numerators and Denominators are simple Terms; which will no longer labour under those difficulties, that in the other form seem'd almost insuperable. First therefore I shall shew how these Reductions are to be perform'd, or how any compound Quantities may be reduced to such simple Terms, especially when the Methods of computing are not obvious. Then I shall apply this Analysis to the Solution of Problems.

3. Reduction by Division and Extraction of Roots will be plain from the following Examples, when you compare like Methods of Operation in Decimal and in Specious Arithmetick.

Examples

Examples of Reduction by Division.

4. The Fraction $\frac{aa}{b+x}$ being proposed, divide aa by $b+x$ in the following manner:

$$\begin{array}{r}
 b+x) aa + 0 \left(\frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4} + \frac{aax^4}{b^5}, \&c. \right. \\
 \underline{aa + \frac{aax}{b}} \\
 0 - \frac{aax}{b} + 0 \\
 \underline{\frac{aax}{b} - \frac{aax^2}{b^2}} \\
 0 + \frac{a^2x^2}{b^2} + 0 \\
 \underline{+ \frac{a^2x^2}{b^2} - \frac{a^2x^3}{b^3}} \\
 0 - \frac{a^2x^3}{b^3} + 0 \\
 \underline{- \frac{a^2x^3}{b^3} + \frac{a^2x^4}{b^4}} \\
 0 + \frac{a^2x^4}{b^4} \&c.
 \end{array}$$

The Quotient therefore is $\frac{aa}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5}, \&c.$ which Series, being infinitely continued, will be equivalent to $\frac{aa}{b+x}$. Or making x the first Term of the Divisor, in this manner, $x+b) aa + 0$ (the Quotient will be $\frac{aa}{x} - \frac{aab}{x^2} + \frac{aab^2}{x^3} - \frac{a^2b^3}{x^4}, \&c.$ found as by the foregoing Process.

5. In like manner the Fraction $\frac{1}{1+x^2}$ will be reduced to $1 - x^2 + x^4 - x^6 + x^8, \&c.$ or to $x^{-2} - x^{-4} + x^{-6} - x^{-8}, \&c.$

6. And the Fraction $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1+x^2-3x}$ will be reduced to $2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}, \&c.$

7. Here it will be proper to observe, that I make use of $x^{-1}, x^{-2}, x^{-3}, x^{-4}, \&c.$ for $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}, \frac{1}{x^4}, \&c.$ of $x^{\frac{1}{2}}, x^{\frac{3}{2}}, x^{\frac{5}{2}}, x^{\frac{7}{2}}, x^{\frac{9}{2}}, \&c.$ for $\sqrt{x}, \sqrt{x^3}, \sqrt{x^5}, \sqrt{x^7}, \sqrt{x^9}, \&c.$ and of $x^{-\frac{1}{2}}, x^{-\frac{3}{2}}, x^{-\frac{5}{2}}, \&c.$ for $\frac{1}{\sqrt{x}}, \frac{1}{\sqrt{x^3}}, \frac{1}{\sqrt{x^5}}, \&c.$ And this by the Rule of Analogy, as may be apprehended from such Geometrical Progressions as these; $x^3, x^{\frac{5}{2}}, x^2, x^{\frac{3}{2}}, x, x^{\frac{1}{2}}, x^0$ (or 1,) $x^{-\frac{1}{2}}, x^{-1}, x^{-\frac{3}{2}}, x^{-2}, \&c.$

8. In the same manner for $\frac{aa}{x} - \frac{aab}{x^2} + \frac{aab^2}{x^3}$, &c. may be wrote $a^2x^{-1} - a^2bx^{-2} + a^2b^2x^{-3}$, &c.

9. And thus instead of $\sqrt{aa - xx}$ may be wrote $\overline{aa - xx}^{\frac{1}{2}}$; and $\overline{aa - xx}^{\frac{1}{2}}$ instead of the Square of $aa - xx$; and $\frac{abb - y^2}{by + yy}^{\frac{1}{3}}$ instead of $\sqrt[3]{\frac{abb - y^2}{by + yy}}$: And the like of others.

10. So that we may not improperly distinguish Powers into Affirmative and Negative, Integral and Fractional.

Examples of Reduction by Extraction of Roots.

11. The Quantity $aa + xx$ being proposed, you may thus extract its Square-Root.

$$aa + xx \left(a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9} - \frac{21x^{12}}{1024a^{11}}, \&c. \right)$$

$$\begin{array}{r} aa \\ 0 + xx \end{array}$$

$$+ xx + \frac{x^4}{4a^2}$$

$$- \frac{x^4}{4a^2}$$

$$+ \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

$$+ \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$$

$$+ \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{x^{10}}{64a^8} + \frac{x^{12}}{256a^{10}}$$

$$- \frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}}, \&c.$$

$$- \frac{5x^8}{64a^6} - \frac{5x^{10}}{128a^8} + \frac{5x^{12}}{512a^{10}}$$

$$+ \frac{7x^{10}}{128a^8} - \frac{7x^{12}}{512a^{10}}, \&c.$$

$$+ \frac{7x^{10}}{128a^8} + \frac{7x^{12}}{256a^{10}}$$

$$- \frac{21x^{12}}{512a^{10}}, \&c.$$

So that the Root is found to be $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$, &c. Where it may be observed, that towards the end of the Operation I neglect all those Terms, whose Dimensions would exceed the Dimensions of the last Term, to which I intend only to continue the Root, suppose to $\frac{x^{12}}{a^{11}}$.

12. Also the Order of the Terms may be inverted in this manner $xx + aa$, in which case the Root will be found to be $x + \frac{aa}{2x} - \frac{a^4}{8x^3} + \frac{a^6}{16x^5} - \frac{5a^8}{128x^7} \&c.$

13. Thus the Root of $aa - xx$ is $a - \frac{xx}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} \&c.$

14. The Root of $x - xx$ is $x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}, \&c.$

15. Of $aa + bx - xx$ is $a + \frac{bx}{2a} - \frac{xx}{2a} - \frac{b^2x^2}{8a^3}, \&c.$

16. And $\sqrt{\frac{1+axx}{1-bxx}}$ is $\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}a^2x^4 + \frac{1}{16}a^3x^6, \&c.}{1 - \frac{1}{2}bx^2 - \frac{1}{8}b^2x^4 - \frac{1}{16}b^3x^6, \&c.}$ and moreover by actually dividing, it becomes

$$\begin{array}{r} 1 + \frac{1}{2}bx^2 + \frac{3}{8}b^2x^4 + \frac{5}{16}b^3x^6, \&c. \\ + \frac{1}{2}a + \frac{1}{4}ab + \frac{3}{16}ab^2 \\ - \frac{1}{8}a^2 - \frac{1}{16}a^2b \\ + \frac{1}{16}a^3 \end{array}$$

17. But these Operations, by due preparation, may very often be abbreviated; as in the foregoing Example to find $\sqrt{\frac{1+axx}{1-bxx}}$, if the Form of the Numerator and Denominator had not been the same, I might have multiply'd each by $\sqrt{1-bxx}$, which would have produced $\frac{\sqrt{1+ax^2-abx^4}}{1-bxx}$, and the rest of the work might have been performed by extracting the Root of the Numerator only, and then dividing by the Denominator.

18. From hence I imagine it will sufficiently appear, by what means any other Roots may be extracted, and how any compound Quantities, however entangled with Radicals or Denominators, (such as $x^3 + \frac{\sqrt{x-\sqrt{1-xx}}}{\sqrt[3]{axx+x^3}} - \frac{\sqrt[5]{x^3+2x^5-x^7}}{\sqrt[3]{x+xx-\sqrt{2x-x^{\frac{2}{3}}}}}$) may be reduced to infinite Series consisting of simple Terms.

Of the Reduction of affected Equations.

19. As to affected Equations, we must be something more particular in explaining how their Roots are to be reduced to such Series as these; because their Doctrine in Numbers, as hitherto deliver'd by Mathematicians, is very perplexed, and incumber'd with superfluous Operations, so as not to afford proper Specimens for performing the Work in Species. I shall therefore first shew how the

Resolu-

Resolution of affected Equations may be compendiously perform'd in Numbers, and then I shall apply the same to Species.

20. Let this Equation $y^3 - 2y - 5 = 0$ be proposed to be resolved, and let 2 be a Number (any how found) which differs from the true Root less than by a tenth part of itself. Then I make $2 + p = y$, and substitute $2 + p$ for y in the given Equation, by which is produced a new Equation $p^3 + 6p^2 + 10p - 1 = 0$, whose Root is to be sought for, that it may be added to the Quote. Thus rejecting $p^3 + 6p^2$ because of its smallness, the remaining Equation $10p - 1 = 0$, or $p = 0,1$, will approach very near to the truth. Therefore I write this in the Quote, and suppose $0,1 + q = p$, and substitute this fictitious Value of p as before, which produces $q^3 + 6,3q^2 + 11,23q + 0,061 = 0$. And since $11,23q + 0,061 = 0$ is near the truth, or $q = -0,0054$ nearly, (that is, dividing $0,061$ by $11,23$, till so many Figures arise as there are places between the first Figures of this, and of the principal Quote exclusively, as here there are two places between 2 and $0,005$) I write $-0,0054$ in the lower part of the Quote, as being negative; and supposing $-0,0054 + r = q$, I substitute this as before. And thus I continue the Operation as far as I please, in the manner of the following Diagram:

$y^3 - 2y - 5 = 0$	$+ 2,10000000$ $- 0,00544852$ $+ 2,09455148, \&c. = y$
$2 + p = y.$ $+ y^3$ $- 2y$ $- 5$	$+ 8 + 12p + 6p^2 + p^3$ $- 4 - 2p$ $- 5$
The Sum	$- 1 + 10p + 6p^2 + p^3$
$0,1 + q = p.$ $+ p^3$ $+ 6p^2$ $+ 10p$ $- 1$	$+ 0,001 + 0,03q + 0,3q^2 + q^3$ $+ 0,06 + 1,2 + 6,$ $+ 1, + 10,$ $- 1,$
The Sum	$0,061 + 11,23q + 6,3q^2 + q^3$
$-0,0054 + r = q.$ q^3 $+ 6,3q^2$ $+ 11,23q$ $+ 0,061$	$- 0,000000157464 + 0,00008748r - 0,0162r^2 + r^3$ $+ 0,000183708 - 0,06804 + 6,3$ $- 0,060642 + 11,23$ $+ 0,061$
The Sum	$+ 0,0005416 + 11,162r$
$-0,00004852 + s = r.$	

21. But the Work may be much abbreviated towards the end by this Method, especially in Equations of many Dimensions. Having first determin'd how far you intend to extract the Root, count so many places after the first Figure of the Coefficient of the last Term but one, of the Equations that result on the right side of the Diagram, as there remain places to be fill'd up in the Quote, and reject the Decimals that follow. But in the last Term the Decimals may be neglected, after so many more places as are the decimal places that are fill'd up in the Quote. And in the antepenultimate Term reject all that are after so many fewer places. And so on, by proceeding Arithmetically, according to that Interval of places: Or, which is the same thing, you may cut off every where so many Figures as in the penultimate Term, so that their lowest places may be in Arithmetical Progression, according to the Series of the Terms, or are to be suppos'd to be supply'd with Cyphers, when it happens otherwise. Thus in the present Example, if I desired to continue the Quote no farther than to the eighth place of Decimals, when I substituted $0,0054 + r$ for q , where four decimal places are compleated in the Quote, and as many remain to be compleated, I might have omitted the Figures in the five inferior places, which therefore I have mark'd or cancell'd by little Lines drawn through them; and indeed I might also have omitted the first Term r^3 , although its Coefficient be $0,99999$. Those Figures therefore being expunged, for the following Operation there arises the Sum $0,0005416 + 11,162r$, which by Division, continued as far as the Term prescribed, gives $-0,00004852$ for r , which compleats the Quote to the Period required. Then subtracting the negative part of the Quote from the affirmative part, there arises $2,09455148$ for the Root of the proposed Equation.

22. It may likewise be observed, that at the beginning of the Work, if I had doubted whether $0,1 + p$ was a sufficient Approximation to the Root, instead of $10p - 1 = 0$, I might have suppos'd that $6p^2 + 10p - 1 = 0$, and so have wrote the first Figure of its Root in the Quote, as being nearer to nothing. And in this manner it may be convenient to find the second, or even the third Figure of the Quote, when in the secondary Equation, about which you are conversant, the Square of the Coefficient of the penultimate Term is not ten times greater than the Product of the last Term multiply'd into the Coefficient of the antepenultimate Term. And indeed you will often save some pains, especially in Equations of many Dimensions, if you seek for all the Figures

to be added to the Quote in this manner; that is, if you extract the lesser Root out of the three last Terms of its secondary Equation: For thus you will obtain, at every time, as many Figures again in the Quote.

23. And now from the Resolution of numeral Equations, I shall proceed to explain the like Operations in Species; concerning which, it is necessary to observe what follows.

24. First, that some one of the specious or literal Coefficients, if there are more than one, should be distinguish'd from the rest, which either is, or may be suppos'd to be, much the least or greatest of all, or nearest to a given Quantity. The reason of which is, that because of its Dimensions continually increasing in the Numerators, or the Denominators of the Terms of the Quote, those Terms may grow less and less, and therefore the Quote may constantly approach to the Root required; as may appear from what is said before of the Species x , in the Examples of Reduction by Division and Extraction of Roots. And for this Species, in what follows, I shall generally make use of x or z ; as also I shall use y, p, q, r, s , &c. for the Radical Species to be extracted.

25. Secondly, when any complex Fractions, or surd Quantities, happen to occur in the proposed Equation, or to arise afterwards in the Process, they ought to be removed by such Methods as are sufficiently known to Analysts. As if we should have $y^3 + \frac{bb}{b-x} y^2 - x^3 = 0$, multiply by $b-x$, and from the Product $by^3 - xy^3 + b^2y^2 - bx^3 + x^4 = 0$ extract the Root y . Or we might suppose $y \times \frac{b}{b-x} = v$, and then writing $\frac{v}{b-x}$ for y , we should have $v^3 + b^2v^2 - b^3x^3 + 3b^2x^4 - 3bx^5 + x^6 = 0$, whence extracting the Root v , we might divide the Quote by $b-x$, in order to obtain y . Also if the Equation $y^3 - xy^{\frac{1}{2}} + x^{\frac{4}{3}} = 0$ were proposed, we might put $y^{\frac{1}{2}} = v$, and $x^{\frac{1}{3}} = z$, and so writing vv for y , and z^3 for x , there will arise $v^6 - z^3v + z^4 = 0$; which Equation being resolved, y and x may be restored. For the Root will be found $v = z + z^3 + 6z^5$, &c. and restoring y and x , we have $y^{\frac{1}{2}} = x^{\frac{1}{3}} + x + 6x^{\frac{5}{3}}$, &c. then squaring, $y = x^{\frac{2}{3}} + 2x^{\frac{4}{3}} + 13x^2$, &c.

26. After the same manner if there should be found negative Dimensions of x and y , they may be removed by multiplying by the same x and y . As if we had the Equation $x^3 + 3x^2y^{-1} - 2x^{-1} - 16y^3 = 0$, multiply by x and y^3 , and there would arise $x^4y^3 + 3x^3y^2 - 2y^3 - 16x = 0$. And if the Equation were $x = \frac{aa}{y} - \frac{2a^3}{y^2} + \frac{3a^4}{y^3}$,
by

by multiplying into y^3 there would arise $xy^3 = a^2y^2 - 2a^2y + 3a^4$. And so of others.

27. Thirdly, when the Equation is thus prepared, the work begins by finding the first Term of the Quote; concerning which, as also for finding the following Terms, we have this general Rule, when the indefinite Species (x or z) is supposed to be small; to which Case the other two Cases are reducible.

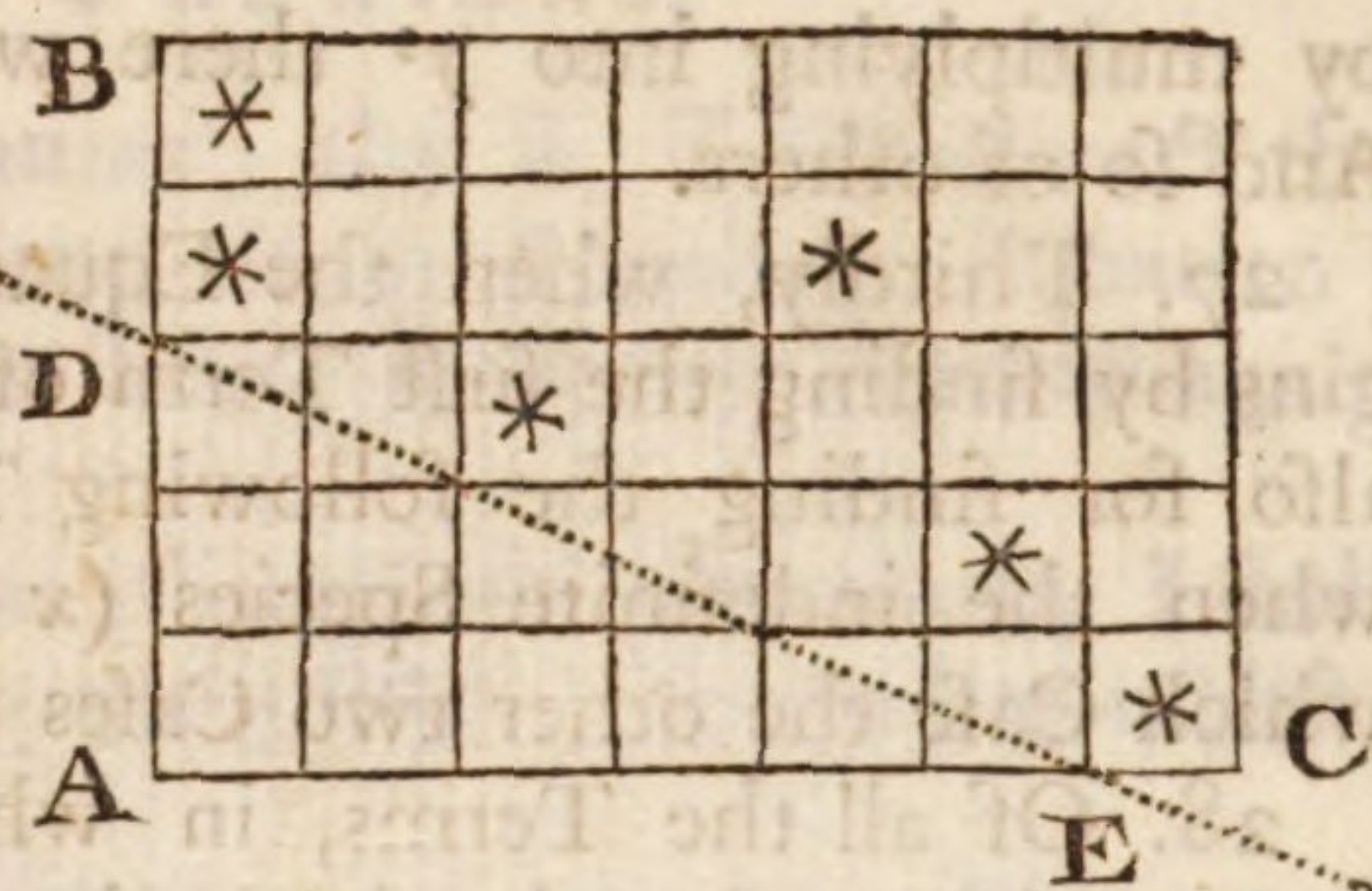
28. Of all the Terms, in which the Radical Species (y, p, q , or r , &c.) is not found, chuse the lowest in respect of the Dimensions of the indefinite Species (x or z , &c.) then chuse another Term in which that Radical Species is found, such as that the Progression of the Dimensions of each of the fore-mentioned Species, being continued from the Term first assumed to this Term, may descend as much as may be, or ascend as little as may be. And if there are any other Terms, whose Dimensions may fall in with this Progression continued at pleasure, they must be taken in likewise. Lastly, from these Terms thus selected, and made equal to nothing, find the Value of the said Radical Species, and write it in the Quote.

29. But that this Rule may be more clearly apprehended, I shall explain it farther by help of the following Diagram. Making a right Angle BAC, divide its sides AB, AC, into equal parts, and raising Perpendiculars, distribute the Angular Space into equal Squares or Parallelograms, which you may conceive to be denominated from the Dimensions of the Species x and y , as they are here inscribed. Then, when any Equation is proposed, mark such of the Parallelograms as correspond to all its Terms, and let a Ruler be apply'd to two, or perhaps more, of the Parallelograms so mark'd, of which let one be the lowest in the left-hand Column at AB, the other touching the Ruler towards the right-hand; and let all the rest, not touching the Ruler, lie above it. Then select those Terms of the Equation which are represented by the Parallelograms that touch the Ruler, and from them find the Quantity to be put in the Quote.

B	x^4	x^4y	x^4y^2	x^4y^3	x^4y^4
	x^3	x^3y	x^3y^2	x^3y^3	x^3y^4
	x^2	x^2y	x^2y^2	x^2y^3	x^2y^4
	x	xy	x^2y	xy^3	xy^4
A	1	y	y^2	y^3	y^4
					C

30. Thus to extract the Root y out of the Equation $y^6 - 5xy^5 + \frac{x^3}{a^2}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$, I mark the Parallelograms belonging

ing to the Terms of this Equation with the Mark *, as you see here done. Then I apply the Ruler DE to the lower of the Parallelograms mark'd in the left-hand Column, and I make it turn round towards the right-hand from the lower to the upper, till it begins



in like manner to touch another, or perhaps more, of the Parallelograms that are mark'd; and I see that the places so touch'd belong to x^3 , x^2y^2 , and y^6 . Therefore from the Terms $y^6 - 7a^2x^2y^2 + 6a^3x^3$, as if equal to nothing, (and moreover, if you please, reduced to $v^6 - 7v^2 + 6 = 0$, by making $y = v\sqrt{ax}$;) I seek the Value of y , and find it to be four-fold, $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$, and $-\sqrt{2ax}$, of which I may take any one for the initial Term of the Quote, according as I design to extract this or that Root of the given Equation.

31. Thus having the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$, I chuse the Terms $-by^2 + 9bx^2$, and thence I obtain $+3x$ for the initial Term of the Quote.

32. And having $y^3 + axy + aay - x^3 - 2a^3 = 0$, I make choice of $y^3 + a^2y - 2a^3$, and its Root $+a$ I write in the Quote.

33. Also having $x^2y^5 - 3c^4xy^2 - c^5x^2 + c^7 = 0$, I select $x^2y^5 + c^7$, which gives $-\sqrt[5]{\frac{c^7}{x^2}}$ for the first Term of the Quote. And the like of others.

34. But when this Term is found, if its Power should happen to be negative, I depress the Equation by the same Power of the indefinite Species, that there may be no need of depressing it in the Resolution; and besides, that the Rule hereafter deliver'd, for the suppression of superfluous Terms, may be conveniently apply'd. Thus the Equation $8z^6y^3 + az^6y^2 - 27a^9 = 0$ being proposed, whose Root is to begin by the Term $\frac{3a^3}{2z^2}$, I depress by z^2 , that it may become $8z^4y^3 + az^4y^2 - 27a^9z^{-2} = 0$, before I attempt the Resolution.

35. The subsequent Terms of the Quotes are derived by the same Method, in the Progress of the Work, from their several secondary Equations, but commonly with less trouble. For the whole affair is perform'd by dividing the lowest of the Terms affected with the indefinitely small Species, (x , x^2 , x^3 , &c.) without the Radical Species, (p , q , r , &c.) by the Quantity with which that radical Species

of one Dimension only is affected, without the other indefinite Species, and by writing the Result in the Quote. So in the following Example, the Terms $\frac{x}{4}$, $\frac{xx}{64a}$, $\frac{131x^3}{512a^2}$, &c. are produced by dividing a^2x , $\frac{1}{16}ax^2$, $\frac{1}{128}x^3$, &c. by $4aa$.

36. These things being premised, it remains now to exhibit the Praxis of Resolution. Therefore let the Equation $y^3 + a^2y + axy - 2a^3 - x^3 = 0$ be proposed to be resolved. And from its Terms $y^3 + a^2y - 2a^3 = 0$, being a fictitious Equation, by the third of the foregoing Premises, I obtain $y - a = 0$, and therefore I write $+a$ in the Quote. Then because $+a$ is not the compleat Value of y , I put $a + p = y$, and instead of y , in the Terms of the Equation written in the Margin, I substitute $a + p$, and the Terms resulting ($p^3 + 3ap^2 + axp$, &c.) I again write in the Margin; from which again, according to the third of the Premises, I select the Terms $+4a^2p + a^2x = 0$ for a fictitious Equation, which giving $p = -\frac{1}{4}x$, I write $-\frac{1}{4}x$ in the Quote. Then because $-\frac{1}{4}x$ is not the accurate Value of p , I put $-\frac{1}{4}x + q = p$, and in the marginal Terms for p I substitute $-\frac{1}{4}x + q$, and the resulting Terms ($q^3 - \frac{3}{4}xq^2 + 3aq^2$, &c.) I again write in the Margin, out of which, according to the foregoing Rule, I again select the Terms $4a^2q - \frac{1}{8}ax^2 = 0$ for a fictitious Equation, which giving $q = \frac{xx}{64a}$, I write $\frac{xx}{64a}$ in the Quote. Again, since $\frac{xx}{64a}$ is not the accurate Value of q , I make $\frac{xx}{64a} + r = q$, and instead of q I substitute $\frac{xx}{64a} + r$ in the marginal Terms. And thus I continue the Process at pleasure, as the following Diagram exhibits to view.

$$\begin{array}{r} 4a^2x \\ + \frac{1}{16}ax^2 \\ + \frac{1}{128}x^3 \\ + \dots \end{array} \div 4aa = \begin{array}{r} a \\ - \frac{1}{4}x \\ + \frac{xx}{64a} \\ + \dots \end{array}$$

$y^3 + a^2y - 2a^3 + axy - x^3 = 0. \quad y = a - \frac{x}{4} + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} \&c.$		
$+a+p=y.$	$+y^3$ $+axy$ $+a^2y$ $-x^3$ $-2a^3$	$+a^3 + 3a^2p + 3ap^2 + p^3$ $+a^2x + axp$ $+a^3 + a^2p$ $-x^3$ $-2a^3$
$-\frac{1}{4}x + q = p.$	$+p^3$ $+3ap^2$ $+axp$ $+4a^2p$ $+a^2x$ $-x^3$	$-\frac{1}{64}x^3 + \frac{3}{16}x^2q - \frac{3}{4}xq^2 + q^3$ $+ \frac{3}{16}ax^2 - \frac{3}{2}axq + 3aq^2$ $-\frac{1}{4}ax^2 + axq$ $-a^2x + 4a^2q$ $+a^2x$ $-x^3$
$+\frac{x^2}{64a} + r = q.$	$+q^3$ $-\frac{3}{4}xq^2$ $+3aq^2$ $+ \frac{3}{16}x^2q$ $-\frac{1}{2}axq$ $+4a^2q$ $-\frac{6}{5}x^3$ $-\frac{1}{16}ax^2$	$*$ $*$ $+ \frac{3x^4}{4096a} * + \frac{3}{32}x^2r + 3ar^2$ $+ \frac{3x^4}{1024a} * + \frac{3}{16}x^2r$ $-\frac{1}{128}x^3 - \frac{1}{2}axr$ $+ \frac{1}{16}ax^2 + 4a^2r$ $-\frac{6}{5}x^3$ $-\frac{1}{16}ax^2$
$+4a^2 - \frac{1}{2}ax + \frac{9}{32}x^2) + \frac{1}{128}x^3 - \frac{15x^4}{4096a} \left(+ \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3} \right.$		

37. If it were required to continue the Quote only to a certain Period, that x , for instance, in the last Term should not ascend beyond a given Dimension; as I substitute the Terms, I omit such as I foresee will be of no use. For which this is the Rule, that after the first Term resulting in the collateral Margin from every Quantity, so many Terms are to be added to the right-hand, as the Index of the highest Power required in the Quote exceeds the Index of that first resulting Term.

38. As in the present Example, if I desired that the Quote, (or the Species x in the Quote,) should ascend no higher than to four Dimensions, I omit all the Terms after x^4 , and put only one after x^3 .

Therefore

Therefore the Terms after the Mark * are to be conceived to be expunged. And thus the Work being continued till at last we come to the Terms $\frac{15x^4}{4096a} - \frac{131x^3}{128} + 4a^2r - \frac{1}{2}axr$, in which p, q, r , or s , &c. representing the Supplement of the Root to be extracted, are only of one Dimension; we may find so many Terms by Division, $(+\frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3})$ as we shall see wanting to compleat the Quote. So that at last we shall have $y = a - \frac{1}{4}x + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c.

39. For the sake of farther Illustration, I shall propose another Example to be resolved. From the Equation $\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y - z = 0$, let the Quote be found only to the fifth Dimension, and the superfluous Terms be rejected after the Mark, &c.

$\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y - z = 0. \quad y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5, \&c.$		
$z + p = y.$	$+\frac{1}{5}y^5$ $-\frac{1}{4}y^4$ $+\frac{1}{3}y^3$ $-\frac{1}{2}y^2$ $+y$ $-z$	$+\frac{1}{5}z^5, \&c.$ $-\frac{1}{4}z^4 - z^3p, \&c.$ $+\frac{1}{3}z^3 + z^2p + zp^2, \&c.$ $-\frac{1}{2}z^2 - zp - \frac{1}{2}p^2$ $+z + p$ $-z$
$\frac{1}{2}z^2 + q = p.$	$+zp^2$ $-\frac{1}{2}p^2$ $-z^3p$ $+z^2p$ $-zp$ $+p$ $+\frac{1}{5}z^5$ $-\frac{1}{4}z^4$ $+\frac{1}{3}z^3$ $-\frac{1}{2}z^2$	$+\frac{1}{4}z^5, \&c.$ $-\frac{1}{8}z^4 - \frac{1}{2}z^2q, \&c.$ $-\frac{1}{2}z^5, \&c.$ $+\frac{1}{2}z^4 + z^2q$ $-\frac{1}{2}z^3 - zq$ $+\frac{1}{2}z^2 + q$ $+\frac{1}{5}z^5$ $-\frac{1}{4}z^4$ $+\frac{1}{3}z^3$ $-\frac{1}{2}z^2$
$1 - z + \frac{1}{2}z^2 - \frac{1}{6}z^3 - \frac{1}{8}z^4 + \frac{1}{20}z^5 (\frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$		

40. And thus if we propose the Equation $\frac{63}{2816}y^{11} + \frac{35}{1152}y^9 + \frac{5}{112}y^7 + \frac{3}{40}y^5 + \frac{1}{8}y^3 + y - z = 0$, to be resolved only to the ninth Dimension of the Quote; before the Work begins we may reject the Term $\frac{63}{2816}y^{11}$; then as we operate we may reject all the Terms beyond z^9 , beyond z^7 we may admit but one, and two only after z^5 ;

z^5 ; because we may observe, that the Quote ought always to ascend by the Interval of two Units, in this manner, $z, z^3, z^5, \&c.$ Then at last we shall have $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9, \&c.$

41. And hence an Artifice is discover'd, by which Equations, tho' affected *in infinitum*, and consisting of an infinite number of Terms, may however be resolved. And that is, before the Work begins all the Terms are to be rejected, in which the Dimension of the indefinitely small Species, not affected by the radical Species, exceeds the greatest Dimension required in the Quote; or from which, by substituting instead of the radical Species, the first Term of the Quote found by the Parallelogram as before, none but such exceeding Terms can arise. Thus in the last Example I should have omitted all the Terms beyond y^9 , though they went on *ad infinitum*. And so in this Equation

$$0 = \begin{cases} -8 + z^2 - 4z^4 + 9z^6 - 16z^8, \&c. \\ +y \text{ in } z^2 - 2z^4 + 3z^6 - 4z^8, \&c. \\ -y^2 \text{ in } z^2 - z^4 + z^6 - z^8, \&c. \\ +y^3 \text{ in } z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6 - \frac{1}{4}z^8, \&c. \end{cases}$$

that the Cubick Root may be extracted only to four Dimensions of z , I omit all the Terms *in infinitum* beyond $+y^3$ in $z^2 - \frac{1}{2}z^4 + \frac{1}{3}z^6$, and all beyond $-y^2$ in $z^2 - z^4 + z^6$, and all beyond $+y$ in $z^2 - 2z^4$, and beyond $-8 + z^2 - 4z^4$. And therefore I assume this Equation only to be resolved, $\frac{1}{3}z^6y^3 - \frac{1}{2}z^4y^3 + z^2y^3 - z^6y^2 + z^4y^2 - z^2y^2 - 2z^4y + z^2y - 4z^4 + z^2 - 8 = 0$. Because $2z^{\frac{2}{3}}$, (the first Term of the Quote,) being substituted instead of y in the rest of the Equation depress'd by $z^{\frac{2}{3}}$, gives every where more than four Dimensions.

42. What I have said of higher Equations may also be apply'd to Quadratics. As if I desired the Root of this Equation

$$0 = \begin{cases} y^2 \\ -y \text{ in } a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} \&c. \\ + \frac{x^4}{4a^2} \end{cases}$$

as far as the Period x^6 , I omit all the Terms *in infinitum*, beyond $-y$ in $a + x + \frac{x^2}{a}$, and assume only this Equation, $y^2 - ay - xy - \frac{x^2}{a^2}y + \frac{x^4}{4a^2} = 0$. This I resolve either in the usual manner, by making

$y = \frac{1}{2}a + \frac{1}{2}x + \frac{x^2}{2a} - \sqrt{\frac{1}{4}a^2 + \frac{1}{2}ax + \frac{3}{4}x^2 + \frac{x^3}{2a}}$; or more expeditiously by the Method of affected Equations deliver'd before, by which we shall have $y = \frac{x^4}{4a^3} - \frac{x^5}{4a^4} *$, where the last Term required vanishes, or becomes equal to nothing.

43. Now after that Roots are extracted to a convenient Period, they may sometimes be continued at pleasure, only by observing the Analogy of the Series. So you may for ever continue this $z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5$, &c. (which is the Root of the infinite Equation $z = y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{4}y^4$, &c.) by dividing the last Term by these Numbers in order 2, 3, 4, 5, 6, &c. And this, $z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$, &c. may be continued by dividing by these Numbers 2×3 , 4×5 , 6×7 , 8×9 , &c. Again, the Series $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$, &c. may be continued at pleasure, by multiplying the Terms respectively by these Fractions, $\frac{1}{2}$, $-\frac{1}{4}$, $-\frac{3}{8}$, $-\frac{5}{8}$, $-\frac{7}{16}$, &c. And so of others.

44. But in discovering the first Term of the Quote, and sometimes of the second or third, there may still remain a difficulty to be overcome. For its Value, sought for as before, may happen to be furd, or the inextricable Root of an high affected Equation. Which when it happens, provided it be not also impossible, you may represent it by some Letter, and then proceed as if it were known. As in the Example $y^3 + axy + a^2y - x^3 - 2a^3 = 0$: If the Root of this Equation $y^3 + a^2y - 2a^3 = 0$, had been furd, or unknown, I should have put any Letter b for it, and then have perform'd the Resolution as follows, suppose the Quote found only to the third Dimension.

$y^3 + aay + axy - 2a^3 - x^3 = 0.$ Make $a^2 + 3b^2 = c^2$, then $y = b - \frac{abx}{c^2} + \frac{a^4x^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8} - \frac{a^5bx^3}{c^8} + \frac{a^5b^3x^3}{c^{10}} \&c.$		
$b + p = y.$	$+ y^3$ $+ axy$ $+ a^2y$ $- x^3$ $- 2a^3$	$+ b^3 + 3b^2p + 3bp^2 + p^3$ $+ abx + axp$ $+ a^2b + a^2p$ $- x^3$ $- 2a^3$
$-\frac{abx}{c^2} + q = p.$	$+ p^3$ $+ 3bp^2$ $+ axp$ $+ c^2p$ $- x^3$ $+ abx$	$-\frac{a^3b^3x^3}{c^6} \&c.$ $+ \frac{3a^2b^3x^2}{c^4} - \frac{6ab^2x}{c^2} q \&c.$ $-\frac{a^2bx^2}{c^2} + axq$ $- abx + c^2q$ $- x^3$ $+ abx$
$c^2 + ax - \frac{6ab^2x}{c^2} \left(\frac{a^4bx^2}{c^4} + x^3 + \frac{a^3b^3x^3}{c^6} \right) \left(\frac{a^4bx^2}{c^6} + \frac{x^3}{c^2} + \frac{a^3b^3x^3}{c^8} \right) \&c.$		

45. Here writing b in the Quote, I suppose $b + p = y$, and then for y I substitute as you see. Whence proceeds $p^3 + 3bp^2$, &c. rejecting the Terms $b^3 + a^2b - 2a^3$, as being equal to nothing: For b is suppos'd to be a Root of this Equation $y^3 + a^2y - 2a^3 = 0$. Then the Terms $3b^2p + a^2p + abx$ give $\frac{-abx}{3b^2 + a^2}$ to be set in the Quote, and $\frac{-abx}{a^2 + 3b^2} + q$ to be substituted for p .

46. But for brevity's sake I write cc for $aa + 3bb$, yet with this caution, that $aa + 3bb$ may be restored, whenever I perceive that the Terms may be abbreviated by it. When the Work is finish'd, I assume some Number for a , and resolve this Equation $y^3 + a^2y - 2a^3 = 0$, as is shewn above concerning Numeral Equations; and I substitute for b any one of its Roots, if it has three Roots. Or rather, I deliver such Equations from Species, as far as I can, especially from the indefinite Species, and that after the manner before insinuated. And for the rest only, if any remain that cannot be expunged, I put Numbers. Thus $y^3 + a^2y - 2a^3 = 0$ will be freed from a , by dividing the Root by a , and it will become $y^3 + y - 2 = 0$, whose Root being found, and multiply'd by a , must be substituted instead of b .

47. Hitherto I have suppos'd the indefinite Species to be little. But if it be suppos'd to approach nearly to a given Quantity, for that indefinitely small difference I put some Species, and that being substituted, I solve the Equation as before. Thus in the Equation $\frac{1}{2}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y + a - x = 0$, it being known or suppos'd that x is nearly of the same Quantity as a , I suppose z to be their difference; and then writing $a + z$ or $a - z$ for x , there will arise $\frac{1}{2}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y \pm z = 0$, which is to be solved as before.

48. But if that Species be suppos'd to be indefinitely great, for its Reciprocal, which will therefore be indefinitely little, I put some Species, which being substituted, I proceed in the Resolution as before. Thus having $y^3 + y^2 + y - x^3 = 0$, where x is known or suppos'd to be very great, for the reciprocally little Quantity $\frac{1}{x}$ I put z , and substituting $\frac{1}{z}$ for x , there will arise $y^3 + y^2 + y - \frac{1}{z^3} = 0$, whose Root is $y = \frac{1}{z} - \frac{1}{3} - \frac{2}{9}z + \frac{7}{81}z^2 + \frac{5}{81}z^3$, &c. where x being restored, if you please, it will be $y = x - \frac{1}{3} + \frac{2}{9x} + \frac{7}{81x^2} + \frac{5}{81x^3}$, &c.

49. If it should happen that none of these Expedients should succeed to your desire, you may have recourse to another. Thus in the Equation $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, whereas the first Term ought to be obtain'd from the Supposition that $y^4 + 2y^2 - 2y + 1 = 0$, which yet admits of no possible Root; you may try what can be done another way. As you may suppose that x is but little different from $+2$, or that $2 + z = x$. Then substituting $2 + z$ instead of x , there will arise $y^4 - z^2y^2 - 3zy^2 - 2y + 1 = 0$, and the Quote will begin from $+1$. Or if you suppose x to be indefinitely great, or $\frac{1}{x} = z$, you will have $y^4 - \frac{y^2}{z^2} + \frac{y^2}{z} + 2y^2 - 2y + 1 = 0$, and $+z$ for the initial Term of the Quote.

50. And thus by proceeding according to several Suppositions, you may extract and express Roots after various ways.

51. If you should desire to find after how many ways this may be done, you must try what Quantities, when substituted for the indefinite Species in the proposed Equation, will make it divisible by y , $+$ or $-$ some Quantity, or by y alone. Which, for Example sake, will happen in the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$,
D
by

by substituting $+a$, or $-a$, or $-2a$, or $-\sqrt[3]{2a^3}$, &c. instead of x . And thus you may conveniently suppose the Quantity x to differ little from $+a$, or $-a$, or $-2a$, or $-\sqrt[3]{2a^3}$, and thence you may extract the Root of the Equation proposed after so many ways. And perhaps also after so many other ways, by supposing those differences to be indefinitely great. Besides, if you take for the indefinite Quantity this or that of the Species which express the Root, you may perhaps obtain your desire after other ways. And farther still, by substituting any fictitious Values for the indefinite Species, such as $ax + bx^2$, $\frac{a}{b+x}$, $\frac{a+cx}{b+x}$, &c. and then proceeding as before in the Equations that will result.

52. But now that the truth of these Conclusions may be manifest; that is, that the Quotes thus extracted, and produced *ad libitum*, approach so near to the Root of the Equation, as at last to differ from it by less than any assignable Quantity, and therefore when infinitely continued, do not at all differ from it: You are to consider, that the Quantities in the left-hand Column of the right-hand side of the Diagrams, are the last Terms of the Equations whose Roots are p, q, r, s , &c. and that as they vanish, the Roots p, q, r, s , &c. that is, the differences between the Quote and the Root sought, vanish at the same time. So that the Quote will not then differ from the true Root. Wherefore at the beginning of the Work, if you see that the Terms in the said Column will all destroy one another, you may conclude, that the Quote so far extracted is the perfect Root of the Equation. But if it be otherwise, you will see however, that the Terms in which the indefinitely small Species is of few Dimensions, that is, the greatest Terms, are continually taken out of that Column, and that at last none will remain there, unless such as are less than any given Quantity, and therefore not greater than nothing when the Work is continued *ad infinitum*. So that the Quote, when infinitely extracted, will at last be the true Root.

53. Lastly, altho' the Species, which for the sake of perspicuity I have hitherto suppos'd to be indefinitely little, should however be suppos'd to be as great as you please, yet the Quotes will still be true, though they may not converge so fast to the true Root. This is manifest from the Analogy of the thing. But here the Limits of the Roots, or the greatest and least Quantities, come to be consider'd. For these Properties are in common both to finite and infinite Equations. The Root in these is then greatest or least, when

when there is the greatest or least difference between the Sums of the affirmative Terms, and of the negative Terms; and is limited when the indefinite Quantity, (which therefore not improperly I suppos'd to be small,) cannot be taken greater, but that the Magnitude of the Root will immediately become infinite, that is, will become impossible.

54. To illustrate this, let ACD be a Semicircle described on the Diameter AD, and BC be an Ordinate. Make $AB = x$, $BC = y$, $AD = a$. Then

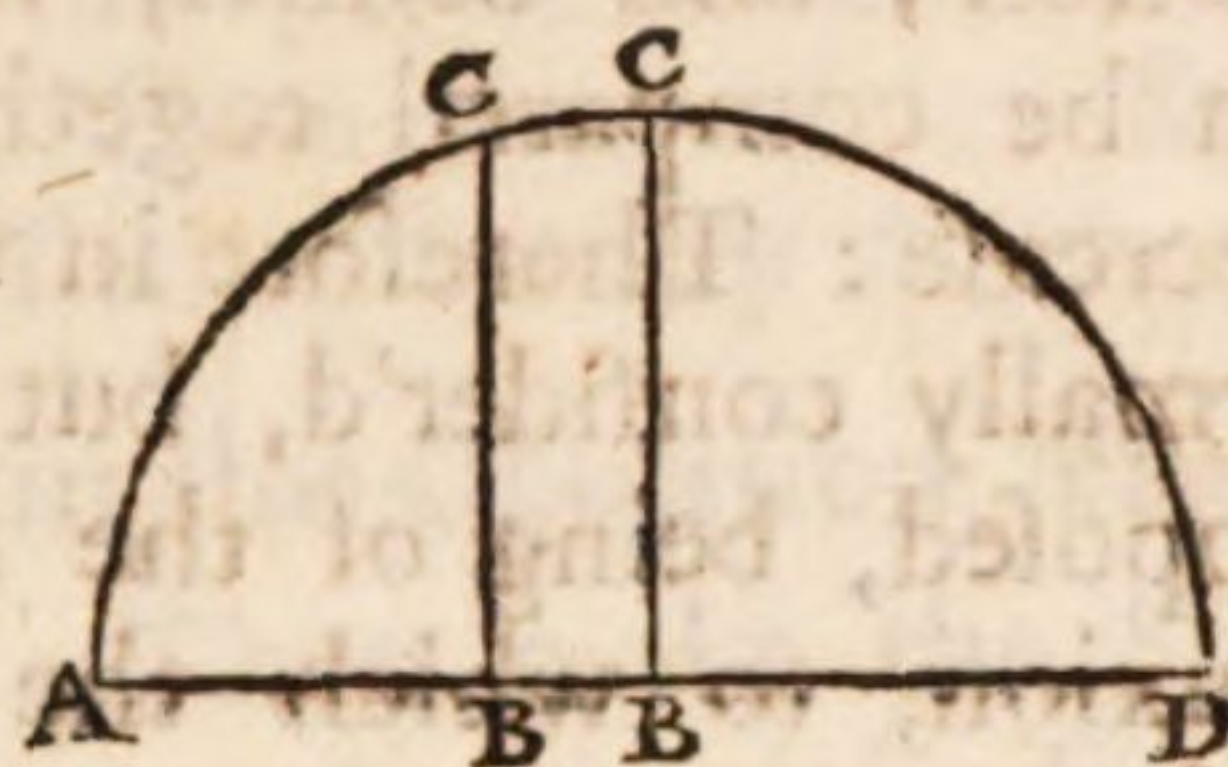
$$y = \sqrt{ax - xx} = \sqrt{ax - \frac{x}{2a} \sqrt{ax} - \frac{x^2}{8a^2} \sqrt{ax} - \frac{x^3}{16a^3} \sqrt{ax}, \&c. \text{ as before.}$$

Therefore BC, or y , then becomes greatest when $\sqrt{ax}$ most exceeds all the Terms

$\frac{x}{2a} \sqrt{ax} + \frac{x^2}{8a^2} \sqrt{ax} + \frac{x^3}{16a^3} \sqrt{ax}, \&c.$ that is, when $x = \frac{1}{2}a$; but it will be terminated when $x = a$. For if we take x greater than

a , the Sum of all the Terms $-\frac{x}{2a} \sqrt{ax} - \frac{x^2}{8a^2} \sqrt{ax} - \frac{x^3}{16a^3} \sqrt{ax}, \&c.$ will be infinite. There is another Limit also, when $x = 0$,

by reason of the impossibility of the Radical $\sqrt{-ax}$; to which Terms or Limits, the Limits of the Semicircle A, B, and D, are correspondent.



Transition to the METHOD OF FLUXIONS.

55. And thus much for the Methods of Computation, of which I shall make frequent use in what follows. Now it remains, that for an Illustration of the Analytick Art, I should give some Specimens of Problems, especially such as the nature of Curves will supply. But first it may be observed, that all the difficulties of these may be reduced to these two Problems only, which I shall propose concerning a Space described by local Motion, any how accelerated or retarded.

56. I. *The Length of the Space described being continually (that is, at all Times) given; to find the Velocity of the Motion at any Time proposed.*

57. II. *The Velocity of the Motion being continually given; to find the Length of the Space described at any Time proposed.*

58. Thus in the Equation $xx = y$, if y represents the Length of the Space at any time described, which (time) another Space x , by increasing with an uniform Celerity $\dot{x}$, measures and exhibits as described:

described: Then $2\dot{x}$ will represent the Celerity by which the Space y , at the same moment of Time, proceeds to be described; and contrary-wise. And hence it is, that in what follows, I consider Quantities as if they were generated by continual Increase, after the manner of a Space, which a Body or Thing in Motion describes.

59. But whereas we need not consider the Time here, any farther than as it is expounded and measured by an equable local Motion; and besides, whereas only Quantities of the same kind can be compared together, and also their Velocities of Increase and Decrease: Therefore in what follows I shall have no regard to Time formally consider'd, but I shall suppose some one of the Quantities proposed, being of the same kind, to be increased by an equable Fluxion, to which the rest may be referr'd, as it were to Time; and therefore, by way of Analogy, it may not improperly receive the name of Time. Whenever therefore the word *Time* occurs in what follows, (which for the sake of perspicuity and distinction I have sometimes used,) by that Word I would not have it understood as if I meant Time in its formal Acceptation, but only that other Quantity, by the equable Increase or Fluxion whereof, Time is expounded and measured.

60. Now those Quantities which I consider as gradually and indefinitely increasing, I shall hereafter call *Fluents*, or *Flowing Quantities*, and shall represent them by the final Letters of the Alphabet v , x , y , and z ; that I may distinguish them from other Quantities, which in Equations are to be consider'd as known and determinate, and which therefore are represented by the initial Letters a , b , c , &c. And the Velocities by which every Fluent is increased by its generating Motion, (which I may call *Fluxions*, or simply Velocities or Celerities,) I shall represent by the same Letters pointed thus $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. That is, for the Celerity of the Quantity v I shall put $\dot{v}$, and so for the Celerities of the other Quantities x , y , and z , I shall put $\dot{x}$, $\dot{y}$, and $\dot{z}$ respectively.

61. These things being premised, I shall now forthwith proceed to the matter in hand; and first I shall give the Solution of the two Problems just now proposed.

P R O B.

P R O B. I.

The Relation of the Flowing Quantities to one another being given, to determine the Relation of their Fluxions.

S O L U T I O N.

1. Dispose the Equation, by which the given Relation is express'd, according to the Dimensions of some one of its flowing Quantities, suppose x , and multiply its Terms by any Arithmetical Progression, and then by $\frac{\dot{x}}{x}$. And perform this Operation separately for every one of the flowing Quantities. Then make the Sum of all the Products equal to nothing, and you will have the Equation required.

2. EXAMPLE I. If the Relation of the flowing Quantities x and y be $x^3 - ax^2 + axy - y^3 = 0$; first dispose the Terms according to x , and then according to y , and multiply them in the following manner.

$$\begin{array}{r|l} \text{Mult. } x^3 & -ax^2 & +axy & -y^3 \\ \text{by } \frac{3\dot{x}}{x} & \cdot \frac{2\dot{x}}{x} & \cdot \frac{\dot{x}}{x} & \cdot 0 \\ \hline \text{makes } 3\dot{x}x^2 & -2a\dot{x}x & +a\dot{x}y & -y^3 \end{array} \quad \begin{array}{r|l} -y^3 & +axy & -ax^2 \\ \frac{3\dot{y}}{y} & \cdot \frac{\dot{y}}{y} & \cdot 0 \\ \hline -3\dot{y}y^2 & +a\dot{y}x & \end{array}$$

The Sum of the Products is $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$, which Equation gives the Relation between the Fluxions $\dot{x}$ and $\dot{y}$. For if you take x at pleasure, the Equation $x^3 - ax^2 + axy - y^3 = 0$ will give y . Which being determined, it will be $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$.

3. Ex. 2. If the Relation of the Quantities x , y , and z , be express'd by the Equation $2y^3 + x^2y - 2cyz + 3yz^2 - z^3 = 0$,

$$\begin{array}{r|l|l} \text{Mult. } 2y^3 & +x^2y & -2cyz & +3yz^2 & -z^3 \\ & -2cz & & & \\ & +3z^2 & & & \\ \text{by } \frac{2\dot{y}}{y} & \cdot 0 & \cdot \frac{\dot{y}}{y} & \cdot \frac{2\dot{x}}{x} & \cdot 0 \\ \hline \text{makes } 4\dot{y}y^2 & * & +\frac{\dot{y}z^3}{y} & 2\dot{x}xy & * & -3\dot{z}z^2 & +6\dot{z}zy & -2c\dot{z}y & * \end{array}$$

Where-

Wherefore the Relation of the Celerities of Flowing, or of the Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, is $4\dot{y}y^2 + \frac{\dot{y}z^3}{y} + 2\dot{x}xy - 3\dot{z}z^2 + 6\dot{z}zy - 2\dot{c}zy = 0$.

4. But since there are here three flowing Quantities, x , y , and z , another Equation ought also to be given, by which the Relation among them, as also among their Fluxions, may be intirely determined. As if it were supposed that $x + y - z = 0$. From whence another Relation among the Fluxions $\dot{x} + \dot{y} - \dot{z} = 0$ would be found by this Rule. Now compare these with the foregoing Equations, by expunging any one of the three Quantities, and also any one of the Fluxions, and then you will obtain an Equation which will intirely determine the Relation of the rest.

5. In the Equation propos'd, whenever there are complex Fractions, or surd Quantities, I put so many Letters for each, and supposing them to represent flowing Quantities, I work as before. Afterwards I suppress and exterminate the assumed Letters, as you see done here.

6. Ex. 3. If the Relation of the Quantities x and y be $yy - aa - x\sqrt{aa - xx} = 0$; for $x\sqrt{aa - xx}$ I write z , and thence I have the two Equations $yy - aa - z = 0$, and $a^2x^2 - x^4 - z^2 = 0$, of which the first will give $2\dot{y}y - \dot{z} = 0$, as before, for the Relation of the Celerities $\dot{y}$ and $\dot{z}$, and the latter will give $2a^2\dot{x}x - 4\dot{x}x^3 - 2\dot{z}z = 0$, or $\frac{a^2\dot{x}x - 2\dot{x}x^3}{z} = \dot{z}$, for the Relation of the Celerities $\dot{x}$ and $\dot{z}$. Now z being expunged, it will be $2\dot{y}y \frac{-a^2x + 2xx^3}{z} = 0$, and then restoring $x\sqrt{aa - xx}$ for z , we shall have $2\dot{y}y \frac{-a^2x + 2xx^3}{\sqrt{aa - xx}} = 0$, for the Relation between $\dot{x}$ and $\dot{y}$, as was required.

7. Ex. 4. If $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay + xx} = 0$, expresses the Relation that is between x and y : I make $\frac{by^3}{a+y} = z$, and $xx\sqrt{ay + xx} = v$, from whence I shall have the three Equations $x^3 - ay^2 + z - v = 0$, $az + yz - by^3 = 0$, and $ax^4y + x^6 - vv = 0$. The first gives $3\dot{x}x^2 - 2a\dot{y}y + \dot{z} - \dot{v} = 0$, the second gives $a\dot{z} + \dot{z}y + y\dot{z} - 3b\dot{y}y^2 = 0$, and the third gives $4a\dot{x}x^3y + 6\dot{x}x^5 + a\dot{y}x^4 - 2\dot{v}v = 0$, for the Relations of the Velocities $\dot{v}$, $\dot{x}$, $\dot{y}$, and $\dot{z}$. But the

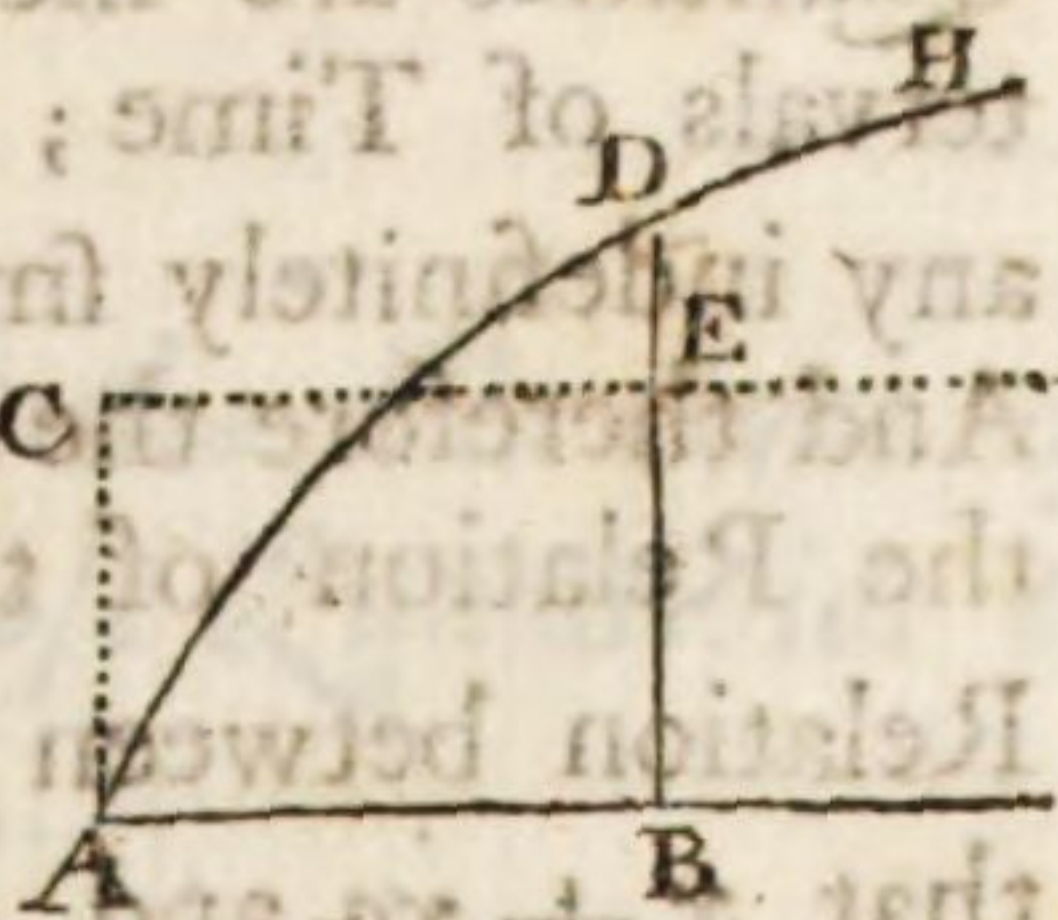
the Values of z and v , found by the second and third Equations, (that is, $\frac{3by^2 - yz}{a+y}$ for z , and $\frac{4axxzy + 6xx^3 + ayx^4}{2v}$ for v) I substitute in the first Equation, and there arises $3xx^2 - 2ayy + \frac{3by^2 - yz}{a+y} - \frac{4axxzy + 6xx^3 + ayx^4}{2v} = 0$. Then instead of z and v restoring their Values $\frac{by^2}{a+y}$ and $xx\sqrt{ay + xx}$, there will arise the Equation sought $3xx^2 - 2ayy + \frac{3aby^2 + 2byy^3 - 4axxy - 6xx^3 - ayxx}{aa + 2ay + yy} - \frac{2\sqrt{ay + xx}}{2\sqrt{ay + xx}} = 0$, by which the Relation of the Velocities x and y will be express'd.

8. After what manner the Operation is to be perform'd in other Cases, I believe is manifest from hence; as when in the Equation propos'd there are found surd Denominators, Cubick Radicals, Radicals within Radicals, as $\sqrt{ax + \sqrt{aa - xx}}$, or any other complicate Terms of the like kind.

9. Furthermore, altho' in the Equation proposed there should be Quantities involved, which cannot be determined or express'd by any Geometrical Method, such as Curvilinear Areas or the Lengths of Curve-lines; yet the Relations of their Fluxions may be found, as will appear from the following Example.

Preparation for EXAMPLE 5.

10. Suppose BD to be an Ordinate at right Angles to AB, and that ADH be any Curve, which is defined by the Relation between AB and BD exhibited by an Equation. Let AB be called x , and the Area of the Curve ADB, apply'd to Unity, be call'd z . Then erect the Perpendicular AC equal to Unity, and thro' C draw CE parallel to AB, and meeting BD in E. Then conceiving these two Superficies ADB and ACEB to be generated by the Motion of the right Line BED; it is manifest that their Fluxions, (that is, the Fluxions of the Quantities $1 \times z$, and $1 \times x$, or of the Quantities z and x ,) are to each other as the generating Lines BD and BE. Therefore $z : x :: BD : BE$ or 1, and therefore $\dot{z} = \dot{x} \times BD$.



11. And hence it is, that z may be involved in any Equation, expressing the Relation between x and any other flowing Quantity y ; and yet the Relation of the Fluxions x and y may however be discover'd.

12. Ex. 5. As if the Equation $zx + axz - y^4 = 0$ were propos'd to express the Relation between x and y , as also $\sqrt{ax - xx} = BD$, for determining a Curve, which therefore will be a Circle. The Equation $zx + axz - y^4 = 0$, as before, will give $2xz + axz + axz - 4yy^3 = 0$, for the Relation of the Celerities $\dot{x}$, $\dot{y}$, and z . And therefore since it is $z = \dot{x} \times BD$ or $= \dot{x} \sqrt{ax - xx}$, substitute this Value instead of it, and there will arise the Equation $2xz + axx \sqrt{ax - xx} + axz - 4yy^3 = 0$, which determines the Relation of the Celerities $\dot{x}$ and $\dot{y}$.

DEMONSTRATION of the Solution.

13. The Moments of flowing Quantities, (that is, their indefinitely small Parts, by the accession of which, in indefinitely small portions of Time, they are continually increased,) are as the Velocities of their Flowing or Increasing.

14. Wherefore if the Moment of any one, as x , be represented by the Product of its Celerity $\dot{x}$ into an indefinitely small Quantity o (that is, by $\dot{x}o$), the Moments of the others y , z , will be represented by $\dot{y}o$, $\dot{z}o$; because $\dot{v}o$, $\dot{x}o$, $\dot{y}o$, and $\dot{z}o$, are to each other as v , x , y , and z .

15. Now since the Moments, as $\dot{x}o$ and $\dot{y}o$, are the indefinitely little accessions of the flowing Quantities x and y , by which those Quantities are increased through the several indefinitely little intervals of Time; it follows, that those Quantities x and y , after any indefinitely small interval of Time, become $x + \dot{x}o$ and $y + \dot{y}o$. And therefore the Equation, which at all times indifferently expresses the Relation of the flowing Quantities, will as well express the Relation between $x + \dot{x}o$ and $y + \dot{y}o$, as between x and y : So that $x + \dot{x}o$ and $y + \dot{y}o$ may be substituted in the same Equation for those Quantities, instead of x and y .

16. Therefore let any Equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and substitute $x + \dot{x}o$ for x , and $y + \dot{y}o$ for y , and there will arise

$$\left. \begin{aligned} & x^3 + 3\dot{x}ox^2 + 3\dot{x}^2oox + \dot{x}^3o^3 \\ & - ax^2 - 2a\dot{x}ox - a\dot{x}^2oo \\ & + axy + a\dot{x}oy + a\dot{y}ox + a\dot{x}\dot{y}oo \\ & - y^3 - 3\dot{y}oy^2 - 3\dot{y}^2ooy - \dot{y}^3o^3 \end{aligned} \right\} = 0.$$

17. Now by Supposition $x^3 - ax^2 + axy - y^3 = 0$, which therefore being expunged, and the remaining Terms being divided by o , there will remain $3\dot{x}x^2 + 3\dot{x}^2ox + \dot{x}^3oo - 2a\dot{x}xx - a\dot{x}^2o + a\dot{x}y + a\dot{y}x + a\dot{x}yo - 3\dot{y}y^2 - 3\dot{y}^2oy - \dot{y}^3oo = 0$. But whereas o is supposed to be infinitely little, that it may represent the Moments of Quantities; the Terms that are multiply'd by it will be nothing in respect of the rest. Therefore I reject them, and there remains $3\dot{x}x^2 - 2a\dot{x}xx + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$, as above in Examp. 1.

18. Here we may observe, that the Terms that are not multiply'd by o will always vanish, as also those Terms that are multiply'd by o of more than one Dimension. And that the rest of the Terms being divided by o , will always acquire the form that they ought to have by the foregoing Rule: Which was the thing to be proved.

19. And this being now shewn, the other things included in the Rule will easily follow. As that in the propos'd Equation several flowing Quantities may be involved; and that the Terms may be multiply'd, not only by the Number of the Dimensions of the flowing Quantities, but also by any other Arithmetical Progressions; so that in the Operation there may be the same difference of the Terms according to any of the flowing Quantities, and the Progression be dispos'd according to the same order of the Dimensions of each of them. And these things being allow'd, what is taught besides in Examp. 3, 4, and 5, will be plain enough of itself.

P R O B. II.

An Equation being proposed, including the Fluxions of Quantities, to find the Relations of those Quantities to one another.

A PARTICULAR SOLUTION.

1. As this Problem is the Converse of the foregoing, it must be solved by proceeding in a contrary manner. That is, the Terms multiply'd by $\dot{x}$ being disposed according to the Dimensions of x ; they must be divided by $\frac{x}{\dot{x}}$, and then by the number of their Dimensions, or perhaps by some other Arithmetical Progression. Then the same work must be repeated with the Terms multiply'd by $\dot{y}$,
E or

or $\dot{z}$, and the Sum resulting must be made equal to nothing, rejecting the Terms that are redundant.

2. EXAMPLE. Let the Equation proposed be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$. The Operation will be after this manner:

Divide	$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y$	Divide	$-3\dot{y}y^2 + a\dot{y}x$
by $\frac{\dot{x}}{x}$. Quot.	$3x^3 - 2ax^2 + a\dot{y}x$	by $\frac{\dot{y}}{y}$. Quot.	$-3y^3 + a\dot{x}y$
Divide by	$3 \quad \cdot \quad 2 \quad \cdot \quad 1.$	Divide by	$3 \quad \cdot \quad 2 \quad \cdot \quad 1.$
Quote	$x^3 - ax^2 + a\dot{y}x$	Quote	$-y^3 + a\dot{x}y$

Therefore the Sum $x^3 - ax^2 + a\dot{y}x - y^3 = 0$, will be the required Relation of the Quantities x and y . Where it is to be observed, that tho' the Term $a\dot{x}y$ occurs twice, yet I do not put it twice in the Sum $x^3 - ax^2 + a\dot{y}x - y^3 = 0$, but I reject the redundant Term. And so whenever any Term recurs twice, (or oftener when there are several flowing Quantities concern'd,) it must be wrote only once in the Sum of the Terms.

3. There are other Circumstances to be observed, which I shall leave to the Sagacity of the Artist; for it would be needless to dwell too long upon this matter, because the Problem cannot always be solved by this Artifice. I shall add however, that after the Relation of the Fluents is obtain'd by this Method, if we can return, by Prob. 1. to the proposed Equation involving the Fluxions, then the work is right, otherwise not. Thus in the Example proposed, after I have found the Equation $x^3 - ax^2 + a\dot{y}x - y^3 = 0$, if from thence I seek the Relation of the Fluxions $\dot{x}$ and $\dot{y}$ by the first Problem, I shall arrive at the proposed Equation $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y - 3\dot{y}y^2 + a\dot{y}x = 0$. Whence it is plain, that the Equation $x^3 - ax^2 + a\dot{y}x - y^3 = 0$ is rightly found. But if the Equation $\dot{x}x - \dot{x}y + a\dot{y} = 0$ were proposed, by the prescribed Method I should obtain this $\frac{1}{2}x^2 - xy + ay = 0$, for the Relation between x and y ; which Conclusion would be erroneous: Since by Prob. 1. the Equation $\dot{x}x - \dot{x}y - \dot{y}x + a\dot{y} = 0$ would be produced, which is different from the former Equation.

4. Having therefore premised this in a perfunctory manner, I shall now undertake the general Solution.

A PREPARATION FOR THE GENERAL SOLUTION.

5. First it must be observed, that in the proposed Equation the Symbols of the Fluxions, (since they are Quantities of a different kind from the Quantities of which they are the Fluxions,) ought to ascend in every Term to the same number of Dimensions: And when it happens otherwise, another Fluxion of some flowing Quantity must be understood to be Unity, by which the lower Terms are so often to be multiply'd, till the Symbols of the Fluxions arise to the same number of Dimensions in all the Terms. As if the Equation $\dot{x} + \dot{x}y\dot{x} - ax\dot{x} = 0$ were proposed, the Fluxion $\dot{z}$ of some third flowing Quantity z must be understood to be Unity, by which the first Term $\dot{x}$ must be multiply'd once, and the last $ax\dot{x}$ twice, that the Fluxions in them may ascend to as many Dimensions as in the second Term $\dot{x}y\dot{x}$: As if the proposed Equation had been derived from this $\dot{x}z + \dot{x}y\dot{x} - az\dot{z}x^2 = 0$, by putting $\dot{z} = 1$. And thus in the Equation $y\dot{x} = yy$, you ought to imagine $\dot{x}$ to be Unity, by which the Term yy is multiply'd.

6. Now Equations, in which there are only two flowing Quantities, which every where arise to the same number of Dimensions, may always be reduced to such a form, as that on one side may be had the Ratio of the Fluxions, (as $\frac{\dot{y}}{\dot{x}}$, or $\frac{\dot{x}}{\dot{y}}$, or $\frac{\dot{z}}{\dot{x}}$, &c.) and on the other side the Value of that Ratio, express'd by simple Algebraic Terms; as you may see here, $\frac{\dot{y}}{\dot{x}} = 2 + 2x - y$. And when the foregoing particular Solution will not take place, it is required that you should bring the Equations to this form.

7. Wherefore when in the Value of that Ratio any Term is denominated by a Compound quantity, or is Radical, or if that Ratio be the Root of an affected Equation; the Reduction must be perform'd either by Division, or by Extraction of Roots, or by the Resolution of an affected Equation, as has been before shewn.

8. As if the Equation $y\dot{a} - y\dot{x} - \dot{x}a + \dot{x}x - \dot{x}y = 0$ were proposed; first by Reduction this becomes $\frac{\dot{y}}{\dot{x}} = 1 + \frac{y}{a-x}$, or $\frac{\dot{x}}{\dot{y}} = \frac{a-x}{a-x+y}$. And in the first Case, if I reduce the Term $\frac{y}{a-x}$, denominated by the compound Quantity $a-x$, to an infinite Series of

E 2
simple

simple Terms $\frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$ &c. by dividing the Numerator y by the Denominator $a - x$, I shall have $\frac{\dot{y}}{x} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$ &c. by the help of which the Relation between x and y is to be determined.

9. So the Equation $\ddot{y}y = \dot{x}\dot{y} + \dot{x}\dot{x}xx$ being given, or $\frac{\ddot{y}y}{xx} = \frac{\dot{y}}{x} + xx$, and by a farther Reduction $\frac{\dot{y}}{x} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + xx}$: I extract the square Root out of the Terms $\frac{1}{4} + xx$, and obtain the infinite Series $\frac{1}{2} + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. which if I substitute for $\sqrt{\frac{1}{4} + xx}$, I shall have $\frac{\dot{y}}{x} = 1 + x^2 - x^4 + 2x^6 - 5x^8$, &c. or $\frac{\dot{y}}{x} = -x^2 + x^4 - 2x^6 + 5x^8$, &c. according as $\sqrt{\frac{1}{4} + xx}$ is either added to $\frac{1}{2}$, or subtracted from it.

10. And thus if the Equation $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2x^3a^3 = 0$ were proposed, or $\frac{\dot{y}^3}{x^3} + ax\frac{\dot{y}}{x} + a^2\frac{\dot{y}}{x} - x^3 - 2a^3 = 0$, I extract the Root of the affected Cubick Equation, and there arises $\frac{\dot{y}}{x} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$ &c. as may be seen before.

11. But here it may be observed, that I look upon those Terms only as compounded, which are compounded in respect of flowing Quantities. For I esteem those as simple Quantities which are compounded only in respect of given Quantities. For they may be reduced to simple Quantities by supposing them equal to other given Quantities. Thus I consider the Quantities $\frac{ax+bx}{c}$, $\frac{x}{a+b}$, $\frac{bcc}{ax+bx}$, $\frac{b^4}{ax^2+bx^2}$, $\sqrt{ax+bx}$, &c. as simple Quantities, because they may may all be reduced to the simple Quantities $\frac{ex}{c}$, $\frac{x}{e}$, $\frac{be^2}{ex}$, $\frac{b^4}{ex^2}$, $\sqrt{ex}$ (or $e^{\frac{1}{2}}x^{\frac{1}{2}}$) &c. by supposing $a + b = e$.

12. Moreover, that the flowing Quantities may the more easily be distinguish'd from one another, the Fluxion that is put in the Numerator of the Ratio, or the Antecedent of the Ratio, may not improperly be call'd the *Relate Quantity*, and the other in the Denominator, to which it is compared, the *Correlate*: Also the
flowing

flowing Quantities may be distinguish'd by the same Names respectively. And for the better understanding of what follows, you may conceive, that the Correlate Quantity is Time, or rather any other Quantity that flows equably, by which Time is expounded and measured. And that the other, or the Relate Quantity, is Space, which the moving Thing, or Point, any how accelerated or retarded, describes in that Time. And that it is the Intention of the Problem, that from the Velocity of the Motion, being given at every Instant of Time, the Space described in the whole Time may be determined.

13. But in respect of this Problem Equations may be distinguish'd into three Orders.

14. First: In which two Fluxions of Quantities, and only one of their flowing Quantities are involved.

15. Second: In which the two flowing Quantities are involved, together with their Fluxions.

16. Third: In which the Fluxions of more than two Quantities are involved.

17. With these Premises I shall attempt the Solution of the Problem, according to these three Cases.

SOLUTION OF CASE I.

18. Suppose the flowing Quantity, which alone is contain'd in the Equation, to be the Correlate, and the Equation being accordingly dispos'd, (that is, by making on one side to be only the Ratio of the Fluxion of the other to the Fluxion of this, and on the other side to be the Value of this Ratio in simple Terms,) multiply the Value of the Ratio of the Fluxions by the Correlate Quantity, then divide each of its Terms by the number of Dimensions with which that Quantity is there affected, and what arises will be equivalent to the other flowing Quantity.

19. So proposing the Equation $\dot{y}y = \dot{x}y + \dot{x}xxx$; I suppose x to be the Correlate Quantity, and the Equation being accordingly reduced, we shall have $\frac{\dot{y}}{x} = 1 + x^2 - x^4 + 2x^6, \&c.$ Now I mul-

tiple the Value of $\frac{\dot{y}}{x}$ into x , and there arises $x + x^3 - x^5 + 2x^7, \&c.$ which Terms I divide severally by their number of Dimensions, and the Result $x + \frac{1}{2}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7, \&c.$ I put $= y$. And by
this

this Equation will be defined the Relation between x and y , as was required.

20. Let the Equation be $\frac{\dot{y}}{x} = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512a^2} \&c.$ there will arise $y = ax - \frac{x^2}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2} \&c.$ for determining the Relation between x and y .

21. And thus the Equation $\frac{\dot{y}}{x} = \frac{1}{x^3} - \frac{1}{x^2} + \frac{a}{x^{\frac{1}{2}}} - x^{\frac{1}{2}} + x^{\frac{3}{2}},$ gives $y = -\frac{1}{2x^2} + \frac{1}{x} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}.$ For multiply the Value of $\frac{\dot{y}}{x}$ into x , and it becomes $\frac{1}{xx} - \frac{1}{x} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}},$ or $x^{-2} - x^{-1} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}},$ which Terms being divided by the number of Dimensions, the Value of y will arise as before.

22. After the same manner the Equation $\frac{\dot{x}}{y} = \frac{2b^2c}{\sqrt{ay^3}} + \frac{3y^2}{a+b} + \sqrt{by+cy},$ gives $x = -\frac{4b^2c}{\sqrt{ay}} + \frac{y^3}{a+b} + \frac{2}{3}\sqrt{by^3+cy^3}.$ For the Value of $\frac{\dot{x}}{y}$ being multiply'd by y , there arises $\frac{2b^2c}{\sqrt{ay}} + \frac{3y^3}{a+b} + \sqrt{by^3+cy^3}$ or $2b^2ca^{-\frac{1}{2}}y^{-\frac{1}{2}} + \frac{3}{a+b}y^3 + \sqrt{b+c} \times y^{\frac{3}{2}}.$ And thence the Value of x results, by dividing by the number of the Dimensions of each Term.

23. And so $\frac{\dot{y}}{x} = x^{\frac{2}{3}},$ gives $y = \frac{3}{5}x^{\frac{5}{3}}.$ And $\frac{\dot{y}}{x} = \frac{ab}{cx^{\frac{1}{3}}},$ gives $y = \frac{3abx^{\frac{2}{3}}}{2c}.$ But the Equation $\frac{\dot{y}}{x} = \frac{a}{x},$ gives $y = \frac{a}{0}.$ For $\frac{a}{x}$ multiply'd into x makes a , which being divided by the number of Dimensions, which is 0, there arises $\frac{a}{0},$ an infinite Quantity for the Value of y .

24. Wherefore, whenever a like Term shall occur in the Value of $\frac{\dot{y}}{x},$ whose Denominator involves the Correlate Quantity of one Dimension only; instead of the Correlate Quantity, substitute the Sum or the Difference between the same and some other given Quantity to be assumed at pleasure. For there will be the same Relation of Flowing, of the Fluents in the Equation so produced, as of the Equation at first proposed; and the infinite Relate Quantity

tity by this means will be diminish'd by an infinite part of itself, and will become finite, but yet consisting of Terms infinite in number.

25. Therefore the Equation $\frac{y}{x} = \frac{a}{b+x}$ being proposed, if for x I write $b+x$, assuming the Quantity b at pleasure, there will arise $\frac{y}{b+x} = \frac{a}{b+x}$; and by Division $\frac{y}{x} = \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4} \&c.$ And now the Rule foregoing will give $y = \frac{ax}{b} - \frac{ax^2}{2b^2} + \frac{ax^3}{3b^3} - \frac{ax^4}{4b^4} \&c.$ for the Relation between x and y .

26. So if you have the Equation $\frac{y}{x} = \frac{2}{1+x} + 3 - xx$, because of the Term $\frac{2}{1+x}$, if you write $1+x$ for x , there will arise $\frac{y}{1+x} = \frac{2}{1+x} + 2 - 2x - xx$. Then reducing the Term $\frac{2}{1+x}$ into an infinite Series $+ 2 - 2x + 2x^2 - 2x^3 + 2x^4, \&c.$ you will have $\frac{y}{1+x} = 4 - 4x + x^2 - 2x^3 + 2x^4, \&c.$ And then according to the Rule $y = 4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5, \&c.$ for the Relation of x and y .

27. And thus if the Equation $\frac{y}{x} = x^{-\frac{1}{2}} + x^{-1} - x^{\frac{1}{2}}$ were proposed; because I here observe the Term x^{-1} (or $\frac{1}{x}$) to be found, I transmute x , by substituting $1-x$ for it, and there arises $\frac{y}{1-x} = \frac{1}{\sqrt{1-x}} + \frac{1}{1-x} - \sqrt{1-x}$. Now the Term $\frac{1}{1-x}$ produces $1 + x + x^2 + x^3, \&c.$ and the Term $\sqrt{1-x}$ is equivalent to $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3, \&c.$ and therefore $\frac{1}{\sqrt{1-x}}$ or $\frac{1}{1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\&c.}$ is the same as $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3, \&c.$ So that when these Values are substituted, I shall have $\frac{y}{1-x} = 1 + 2x + \frac{3}{2}x^2 + \frac{27}{16}x^3, \&c.$ And then by the Rule $y = x + x^2 + \frac{1}{2}x^3 + \frac{27}{8}x^4, \&c.$ And so of others.

28. Also in other Cases the Equation may sometimes be conveniently reduced, by such a Transmutation of the flowing Quantity.

As if this Equation were proposed $\frac{y}{x} = \frac{c^2x}{c^2 - 3c^2x + 3cx^2 - x^3}$; instead of

of x I write $c - x$, and then I shall have $\frac{\dot{y}}{x} = \frac{c^3 - c^2 x}{x^3}$ Or $\frac{c^3}{x^3} - \frac{c^2}{x^2}$; and then by the Rule $y = -\frac{c^3}{2xx} + \frac{c^2}{x}$. But the use of such Transmutations will appear more plainly in what follows.

SOLUTION OF CASE II.

29. PREPARATION. And so much for Equations that involve only one Fluent. But when each of them are found in the Equation, first it must be reduced to the Form prescribed, by making, that on one side may be had the Ratio of the Fluxions, equal to an aggregate of simple Terms on the other side.

30. And besides, if in the Equations so reduced there be any Fractions denominated by the flowing Quantity, they must be freed from those Denominators, by the above-mentioned Transmutation of the flowing Quantity.

31. So the Equation $yax - xxy - aax = 0$ being proposed, or $\frac{\dot{y}}{x} = \frac{y}{a} + \frac{a}{x}$; because of the Term $\frac{a}{x}$, I assume b at pleasure, and for x I either write $b + x$, or $b - x$, or $x - b$. As if I should write $b + x$, it will become $\frac{\dot{y}}{x} = \frac{y}{a} + \frac{a}{b+x}$. And then the Term $\frac{a}{b+x}$ being converted by Division into an infinite Series, we shall have $\frac{\dot{y}}{x} = \frac{y}{a} + \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$, &c.

32. And after the same manner the Equation $\frac{\dot{y}}{x} = 3y - 2x + \frac{x}{y} - \frac{2y}{xx}$ being proposed; if, by reason of the Terms $\frac{x}{y}$ and $\frac{2y}{xx}$, I write $1 - y$ for y , and $1 - x$ for x , there will arise $\frac{\dot{y}}{x} = 1 - 3y + 2x + \frac{1-x}{1-y} + \frac{2y-2}{1-2x+x^2}$. But the Term $\frac{1-x}{1-y}$ by infinite Division gives $1 - x + y - xy + y^2 - xy^2 + y^3 - xy^3$, &c. and the Term $\frac{2y-2}{1-2x+x^2}$ by a like Division gives $2y - 2 + 4xy - 4x + 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4$, &c. Therefore $\frac{\dot{y}}{x} = -3x + 3xy + y^2 - xy^2 + y^3 - xy^3$, &c. $+ 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10x^4y - 10x^4$, &c.

33. RULE. The Equation being thus prepared, when need requires, dispose the Terms according to the Dimensions of the flowing Quantities, by setting down first those that are not affected by the Relate Quantity, then those that are affected by its least Dimension, and so on. In like manner also dispose the Terms in each of these Classes according to the Dimensions of the other Correlate Quantity, and those in the first Class, (or such as are not affected by the Relate Quantity,) write in a collateral order, proceeding towards the right hand, and the rest in a descending Series in the left-hand Column, as the following Diagrams indicate. The work being thus prepared, multiply the first or the lowest of the Terms in the first Class by the Correlate Quantity, and divide by the number of Dimensions, and put this in the Quote for the initial Term of the Value of the Relate Quantity. Then substitute this into the Terms of the Equation that are disposed in the left-hand Column, instead of the Relate Quantity, and from the next lowest Terms you will obtain the second Term of the Quote, after the same manner as you obtain'd the first. And by repeating the Operation you may continue the Quote as far as you please. But this will appear plainer by an Example or two.

34. EXAMP. I. Let the Equation $\frac{y}{x} = 1 - 3x + y + x^2 + xy$ be proposed, whose Terms $1 - 3x + x^2$, which are not affected by the Relate Quantity y , you see dispos'd collaterally in the up-

	$+ 1 - 3x + xx$
$+ y$	$* + x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5, \&c.$
$+ xy$	$* * + xx - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6, \&c.$
The Sum	$1 - 2x + xx - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{4}{30}x^5, \&c.$
$y =$	$x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6, \&c.$

permost Row, and the rest y and xy in the left-hand Column. And first I multiply the initial Term 1 into the Correlate Quantity x , and it makes x , which being divided by the number of Dimensions 1 , I place it in the Quote under-written. Then substituting this Term instead of y in the marginal Terms $+ y$ and $+ xy$, I have $+ x$ and $+ xx$, which I write over against them to the right hand. Then from the rest I take the lowest Terms $- 3x$ and $+ x$, whose aggregate $- 2x$ multiply'd into x becomes $- 2xx$, and
F being

being divided by the number of Dimensions 2, gives $-xx$ for the second Term of the Value of y in the Quote. Then this Term being likewise assumed to compleat the Value of the Marginals $+y$ and $+xy$, there will arise also $-xx$ and $-x^3$, to be added to the Terms $+x$ and $+xx$ that were before inserted. Which being done, I again assume the next lowest Terms $+xx$, $-xx$, and $+xx$, which I collect into one Sum xx , and thence I derive (as before) the third Term $+\frac{1}{3}x^3$, to be put in the Value of y . Again, taking this Term $+\frac{1}{3}x^3$ into the Values of the marginal Terms, from the next lowest $+\frac{1}{3}x^3$ and $-x^3$ added together, I obtain $-\frac{1}{6}x^4$ for the fourth Term of the Value of y . And so on *in infinitum*.

35. EXAMP. 2. In like manner if it were required to determine the Relation of x and y in this Equation, $\frac{y}{x} = 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4} + \frac{x^4y}{a^5} + \dots$, which Series is supposed to proceed *ad infinitum*; I put 1 in the beginning, and the other Terms in the left-hand Column, and then pursue the work according to the following Diagram.

	+ 1
$+\frac{y}{a}$	* $+\frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{2a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}, \&c.$
$+\frac{xy}{a^2}$	* * $+\frac{x^2}{a^2} + \frac{x^3}{2a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}, \&c.$
$+\frac{x^2y}{a^3}$	* * * $+\frac{x^3}{a^3} + \frac{x^4}{2a^4} + \frac{x^5}{2a^5}, \&c.$
$+\frac{x^3y}{a^4}$	* * * * $+\frac{x^4}{a^4} + \frac{x^5}{2a^5}, \&c.$
$+\frac{x^4y}{a^5}$	* * * * * $+\frac{x^5}{a^5}, \&c.$
&c.	
Sum	$1 + \frac{x}{a} + \frac{3x^2}{2a^2} + \frac{2x^3}{a^3} + \frac{5x^4}{2a^4} + \frac{3x^5}{a^5}, \&c.$
$y =$	$x + \frac{x^2}{2a} + \frac{x^3}{2a^2} + \frac{x^4}{2a^3} + \frac{x^5}{2a^4} + \frac{x^6}{2a^5}, \&c.$

36. As I here proposed to extract the Value of y as far as six Dimensions of x only; for that reason I omit all the Terms in the Operation which I foresee will contribute nothing to my purpose, as is intimated by the Mark, &c. which I have subjoin'd to the Series that are cut off.

37. EXAMP. 3. In like manner if this Equation were proposed $\frac{y}{x} = -3x + 3xy + y^2 - xy^2 + y^3 - xy^3 + y^4 - xy^4, \&c. + 6x^2y - 6x^2 + 8x^3y - 8x^3 + 10xy^4 - 10x^4, \&c.$ and it is intended to extract the Value of y as far as seven Dimensions of x . I place the Terms in order, according to the following Diagram, and I work as before, only with this exception, that since in the left-hand Column y is not only of one, but also of two and three Dimensions; (or of more than three, if I intended to produce the Value of y beyond the degree of x^7 ;) I subjoin the second and third Powers of the Value of y , so far gradually produced, that when they are substituted by degrees to the right-hand, in the Values of the Marginals

	$-3x - 6x^2 - 8x^3 - 10x^4 - 12x^5 - 14x^6, \&c.$
$+ 3xy$	$* \quad * \quad -\frac{9}{2}x^3 - 6x^4 - \frac{75}{8}x^5 - \frac{273}{20}x^6, \&c.$
$+ 6x^2y$	$* \quad * \quad * \quad -9x^4 - 12x^5 - \frac{75}{4}x^6, \&c.$
$+ 8x^3y$	$* \quad * \quad * \quad * \quad -12x^5 - 16x^6, \&c.$
$+ 10x^4y$	$* \quad * \quad * \quad * \quad * \quad -15x^6, \&c.$
$\&c.$	
$+ y^2$	$* \quad * \quad * \quad +\frac{9}{4}x^4 + 6x^5 + \frac{107}{8}x^6, \&c.$
$- xy^2$	$* \quad * \quad * \quad * \quad -\frac{9}{4}x^5 - 6x^6, \&c.$
$\&c.$	
$+ y^3$	$* \quad * \quad * \quad * \quad * \quad -\frac{27}{8}x^6, \&c.$
Sum	$-3x - 6x^2 - \frac{25}{2}x^3 - \frac{91}{4}x^4 - \frac{333}{8}x^5 - \frac{367}{5}x^6, \&c.$
$y =$	$-\frac{3}{2}x^2 - 2x^3 - \frac{25}{8}x^4 - \frac{91}{20}x^5 - \frac{111}{16}x^6 - \frac{367}{35}x^7, \&c.$
$y^2 =$	$+\frac{9}{4}x^4 + 6x^5 + \frac{107}{8}x^6, \&c.$
$y^3 =$	$-\frac{27}{8}x^6, \&c.$

to the left, Terms may arise of so many Dimensions as I observe to be required for the following Operation. And by this Method there arises at length $y = -\frac{3}{2}x^2 - 6x^3 - \frac{25}{8}x^4, \&c.$ which is the

Equation required. But whereas this Value is negative, it appears that one of the Quantities x or y decreases, while the other increases. And the same thing is also to be concluded, when one of the Fluxions is affirmative, and the other negative.

38. EXAMP. 4. You may proceed in like manner to resolve the Equation, when the Relate Quantity is affected with fractional Dimensions. As if it were proposed to extract the Value of x from this Equation, $\frac{x}{y} = \frac{1}{2}y - 4y^2 + 2yx^{\frac{1}{2}} - \frac{4}{3}x^2 + 7y^{\frac{5}{2}} + 2y^3$, in

	$+\frac{1}{2}y$	*	$-4y^2$	+	$7y^{\frac{5}{2}}$	+	$2y^3$	
$2yx^{\frac{1}{2}}$	*	*	$+y^2$	*	$-2y^3$	+	$4y^{\frac{7}{2}}$	$-2y^4, \&c.$
$-\frac{4}{3}x^2$	*	*	*	*	*	*	$-\frac{1}{3}y^4$	$\&c.$
Sum	$+\frac{1}{2}y$	*	$-3y^2$	+	$7y^{\frac{5}{2}}$	*	$+4y^{\frac{7}{2}}$	$-\frac{4}{3}y^4, \&c.$
$x =$	$+\frac{1}{4}y^2$	$-y^3$	$+2y^{\frac{7}{2}}$	*	$+\frac{8}{9}y^{\frac{9}{2}}$	$-\frac{4}{15}y^5, \&c.$		
$x^{\frac{1}{2}} =$	$+\frac{1}{2}y$	$-y^2$	$+2y^{\frac{5}{2}}$	$-y^3, \&c.$				
$x^2 =$	$\frac{1}{16}y^4, \&c.$							

which x in the Term $2yx^{\frac{1}{2}}$ (or $2y\sqrt{x}$) is affected with the Fractional Dimension $\frac{1}{2}$. From the Value of x I derive by degrees the Value of $x^{\frac{1}{2}}$, (that is, by extracting its square-Root,) as may be observed in the lower part of this Diagram; that it may be inserted and transfer'd gradually into the Value of the marginal Term $2yx^{\frac{1}{2}}$. And so at last I shall have the Equation $x = \frac{1}{4}y^2 - y^3 + 2y^{\frac{7}{2}} + \frac{8}{9}y^{\frac{9}{2}} - \frac{4}{15}y^5, \&c.$ by which x is express'd indefinitely in respect of y . And thus you may operate in any other case whatsoever.

39. I said before, that these Solutions may be perform'd by an infinite variety of ways. This may be done if you assume at pleasure not only the initial quantity of the upper Series, but any other given quantity for the first Term of the Quote, and then you may proceed as before. Thus in the first of the preceding Examples, if you assume 1 for the first Term of the Value of y , and substitute it for y in the marginal Terms $+y$ and $+xy$, and pursue the rest of the Operation as before, (of which I have here given a

	$+ 1 - 3x + xx$
$+ y$	$+ 1 + 2x \quad * \quad + x^3 + \frac{1}{4}x^4, \&c.$
$+ xy$	$* + x + 2x^2 \quad * \quad + x^4, \&c.$
Sum	$+ 2 \quad * \quad + 3x^2 + x^3 + \frac{5}{4}x^4, \&c.$
$y =$	$1 + 2x \quad * \quad + x^3 + \frac{1}{4}x^4 + \frac{1}{4}x^5, \&c.$

Specimen,) another Value of y will arise, $1 + 2x + x^3 + \frac{1}{4}x^4, \&c.$ And thus another and another Value may be produced, by assuming 2, or 3, or any other number for its first Term. Or if you make use of any Symbol, as a , to represent the first Term indefinitely, by the same method of Operation, (which I shall here set down,) you will find $y = a + x + ax - xx + axx + \frac{1}{3}x^3 + \frac{2}{3}ax^3, \&c.$ which being found, for a you may substitute 1, 2, 0, $\frac{1}{4}$, or any other Number, and thereby obtain the Relation between x and y an infinite variety of ways.

	$+ 1 - 3x + xx$
$+ y$	$+ a + x - xx + \frac{1}{3}x^3, \&c.$
$+ xy$	$+ ax + ax^2 + \frac{2}{3}ax^3, \&c.$
	$* + ax + x^2 - x^3, \&c.$
	$+ ax^2 + ax^3, \&c.$
Sum	$+ 1 - 2x + x^2 - \frac{2}{3}x^3, \&c.$
	$+ a + 2ax + 2ax^2 + \frac{5}{3}ax^3, \&c.$
$y =$	$a + x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4, \&c.$
	$+ ax + ax^2 + \frac{2}{3}ax^3 + \frac{5}{12}ax^4, \&c.$

40. And it is to be observed, that when the Quantity to be extracted is affected with a Fractional Dimension, (as you see in the fourth of the preceding Examples,) then it is convenient to take Unity, or some other proper Number, for its first Term. And indeed this is necessary, when to obtain the Value of that fractional Dimension, the Root cannot otherwise be extracted, because of the negative Sign; as also when there are no Terms to be disposed in the first or capital Class, from which that initial Term may be deduced.

41. And thus at last I have compleated this most troublesome and of all others most difficult Problem, when only two flowing Quantities, together with their Fluxions, are comprehended in an Equation. But besides this general Method, in which I have taken in all the Difficulties, there are others which are generally shorter, by which the Work may often be eased; to give some Specimens of which, *ex abundanti*, perhaps will not be disagreeable to the Reader.

42. I. If it happen that the Quantity to be resolved has in some places negative Dimensions, it is not of absolute necessity that therefore the Equation should be reduced to another form. For thus the Equation $\dot{y} = \frac{1}{y} - xx$ being proposed, where y is of one negative Dimension, I might indeed reduce it to another Form, as by writing $1 + y$ for y ; but the Resolution will be more expedite as you have it in the following Diagram.

	* * — xx
$\frac{1}{y}$	$1 - x + \frac{3}{2}xx, \&c.$
Sum	$1 - x + \frac{1}{2}xx$
$y =$	$1 + x - \frac{1}{2}xx + \frac{1}{8}x^3, \&c.$
$\frac{1}{y} =$	$1 - x + \frac{3}{2}xx, \&c.$

43. Here assuming 1 for the initial Term of the Value of y , I extract the rest of the Terms as before, and in the mean time I deduce from thence, by degrees, the Value of $\frac{1}{y}$ by Division, and insert it in the Value of the marginal Term.

44. II. Neither is it necessary that the Dimensions of the other flowing Quantity should be always affirmative. For from the Equation $\dot{y} = 3 + 2y - \frac{yy}{x}$, without the prescribed Reduction of the Term $\frac{yy}{x}$, there will arise $y = 3x - \frac{3}{2}xx + 2x^3, \&c.$

45. And from the Equation $\dot{y} = -y + \frac{1}{x} - \frac{1}{xx}$, the Value of y will be found $y = \frac{1}{x}$, if the Operation be perform'd after the Manner of the following Specimen.

$$- \frac{1}{xx}$$

	$-\frac{1}{xx} + \frac{1}{x}$
$-y$	$-\frac{1}{x}$
Sum	$-\frac{1}{xx} \quad 0$
$y =$	$\frac{1}{x}$

46. Here we may observe by the way, that among the infinite manners by which any Equation may be resolved, it often happens that there are some, that terminate at a finite Value of the Quantity to be extracted, as in the foregoing Example. And these are not difficult to find, if some Symbol be assumed for the first Term. For when the Resolution is perform'd, then some proper Value may be given to that Symbol, which may render the whole finite.

47. III. Again, if the Value of y is to be extracted from this Equation $y = \frac{y}{2x} + 1 - 2x + \frac{1}{2}xx$, it may be done conveniently enough, without any Reduction of the Term $\frac{y}{2x}$, by supposing (after the manner of Analysts,) that to be given which is required. Thus for the first Term of the Value of y I put $2ex$, taking $2e$ for the numeral Coefficient which is yet unknown. And substituting $2ex$ instead of y , in the marginal Term, there arises e , which I write on the right-hand; and the Sum $1 + e$ will give $x + ex$ for the same first Term of the Value of y , which I had first represented by the Term $2ex$. Therefore I make $2ex = x + ex$, and thence I deduce $e = \frac{1}{2}$. So that the first Term $2ex$ of the Value of y is x . After the same manner I make use of the fictitious Term $2fx^2$ to represent the second Term of the Value of y , and thence at last I derive $-\frac{2}{3}$ for the Value of f , and therefore that second Term is $-\frac{4}{3}xx$. And so the fictitious Coefficient g in the third Term will give $\frac{1}{6}$, and h in the fourth Term will be 0. Wherefore since there are no other Terms remaining, I conclude the work is finish'd, and that the Value of y is exactly $x - \frac{4}{3}x^2 + \frac{1}{6}x^3$. See the Operation in the following Diagram.

	$1 - 2x + \frac{1}{2}xx$
$\frac{y}{2x}$	$e + fx + gxx + bx^3$
Sum	$+1 - 2x + \frac{1}{2}xx$ $+e + fx + gxx + bx^3$
Hypothetically $y =$	$2ex + 2fx^2 + 2gx^3 + 2bx^4$
Consequentially $y =$	$+x - x^2 + \frac{1}{6}x^3 + \frac{1}{4}bx^4$ $+ex + \frac{1}{2}fx^2 + \frac{1}{3}gx^3$
Real Value $y =$	$2x - \frac{4}{3}x^2 + \frac{1}{3}x^3$

48. Much after the same manner, if it were $y = \frac{3y}{4x}$; suppose $y = ex^s$, where e denotes the unknown Coefficient, and s the number of Dimensions, which is also unknown. And ex^s being substituted for y , there will arise $y = \frac{3ex^{s-1}}{4}$, and thence again $y = \frac{3ex^s}{4}$. Compare these two Values of y , and you will find $\frac{3e}{4} = e$, and therefore $s = \frac{1}{4}$, and e will be indefinite. Therefore assuming e at pleasure, you will have $y = ex^{\frac{1}{4}}$.

49. IV. Sometimes also the Operation may be begun from the highest Dimension of the equable Quantity, and continually proceed to the lower Powers. As if this Equation were given, $y = \frac{y}{xx} + \frac{1}{xx} + 3 + 2x - \frac{4}{x}$, and we would begin from the highest Term $2x$, by disposing the capital Series in an order contrary to the foregoing; there will arise at last $y = xx + 4x - \frac{1}{x}$, &c. as may be seen in the form of working here set down.

	$+ 2x + 3 - \frac{4}{x} + \frac{1}{xx}$
$+ \frac{y}{xx}$	$* + 1 + \frac{4}{x} * - \frac{1}{x^3} + \frac{1}{2x^4}, \&c.$
Sum	$+ 2x + 4 * + \frac{1}{xx} - \frac{1}{x^3} + \frac{1}{2x^4}, \&c.$
$y =$	$xx + 4x * - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3}, \&c.$

50. And here it may be observed by the way, that as the Operation proceeded, I might have inserted any given Quantity between the Terms $4x$ and $-\frac{1}{x}$, for the intermediate Term that is deficient, and so the Value of y might have been exhibited an infinite variety of ways.

51. V. If there are besides any fractional Indices of the Dimensions of the Relate Quantity, they may be reduced to Integers by supposing that Quantity, which is affected by its fractional Dimension, to be equal to any third Fluent; and then by substituting that Quantity, as also its Fluxion, arising from that fictitious Equation, instead of the Relate Quantity and its Fluxion.

52. As if the Equation $\dot{y} = 3xy^{\frac{2}{3}} + y$ were proposed, where the Relate Quantity is affected with the fractional Index $\frac{2}{3}$ of its Dimension; a Fluent z being assumed at pleasure, suppose $y^{\frac{1}{3}} = z$, or $y = z^3$; the Relation of the Fluxions, by Prob. I. will be $\dot{y} = 3z\dot{z}^2$. Therefore substituting $3z\dot{z}^2$ for $\dot{y}$, as also z^3 for y , and $\dot{z}^2$ for $y^{\frac{2}{3}}$, there will arise $3z\dot{z}^2 = 3xz^2 + z^3$, or $\dot{z} = x + \frac{1}{3}z$, where z performs the office of the Relate Quantity. But after the Value of z is extracted, as $z = \frac{1}{2}x^2 + \frac{x^3}{18} + \frac{x^4}{216} + \frac{x^5}{3240}$, &c. instead of z restore $y^{\frac{1}{3}}$, and you will have the desired Relation between x and y ; that is, $y^{\frac{1}{3}} = \frac{1}{2}x^2 + \frac{1}{18}x^3 + \frac{1}{216}x^4$, &c. and by Cubing each side, $y = \frac{1}{8}x^6 + \frac{1}{4}x^7 + \frac{1}{8}x^8$, &c.

53. In like manner if the Equation $\dot{y} = \sqrt{4y} + \sqrt{xy}$ were given, or $\dot{y} = 2y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}$; I make $z = y^{\frac{1}{2}}$, or $zz = y$, and thence by Prob. I. $2z\dot{z} = \dot{y}$, and by consequence $2z\dot{z} = 2z + x^{\frac{1}{2}}z$, or $\dot{z} = 1 + \frac{1}{2}x^{\frac{1}{2}}$. Therefore by the first Case of this 'tis $z = x + \frac{1}{3}x^{\frac{3}{2}}$, or $y^{\frac{1}{2}} = x + \frac{1}{3}x^{\frac{3}{2}}$, then by squaring each side, $y = xx + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. But if you should desire to have the Value of y exhibited an infinite number of ways, make $z = c + x + \frac{1}{3}x^{\frac{3}{2}}$, assuming any initial Term c , and it will be zz , that is y , $= c^2 + 2cx + \frac{2}{3}cx^{\frac{3}{2}} + x^2 + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. But perhaps I may seem too minute, in treating of such things as will but seldom come into practice.

SOLUTION OF CASE III.

54. The Resolution of the Problem will soon be dispatch'd, when the Equation involves three or more Fluxions of Quantities. For
G
between

between any two of those Quantities any Relation may be assumed, when it is not determined by the State of the Question, and the Relation of their Fluxions may be found from thence; so that either of them, together with its Fluxion, may be exterminated. For which reason if there are found the Fluxions of three Quantities, only one Equation need to be assumed, two if there be four, and so on; that the Equation propos'd may finally be transform'd into another Equation, in which only two Fluxions may be found. And then this Equation being resolved as before, the Relations of the other Quantities may be discover'd.

55. Let the Equation propos'd be $2\dot{x} - \dot{z} + y\dot{x} = 0$; that I may obtain the Relation of the Quantities x , y , and z , whose Fluxions $\dot{x}$, y , and $\dot{z}$ are contained in the Equation; I form a Relation at pleasure between any two of them, as x and y , supposing that $x = y$, or $2y = a + z$, or $x = yy$, &c. But suppose at present $x = yy$, and thence $\dot{x} = 2y\dot{y}$. Therefore writing $2y\dot{y}$ for $\dot{x}$, and $y\dot{y}$ for $\dot{x}$, the Equation propos'd will be transform'd into this: $4y\dot{y} - \dot{z} + y\dot{y} = 0$. And thence the Relation between y and z will arise, $2y\dot{y} + \frac{1}{3}\dot{y}^3 = \dot{z}$. In which if x be written for yy , and $x^{\frac{3}{2}}$ for y^3 , we shall have $2\dot{x} + \frac{1}{3}\dot{x}^{\frac{3}{2}} = \dot{z}$. So that among the infinite ways in which x , y , and z may be related to each other, one of them is here found, which is represented by these Equations, $x = yy$, $2y^2 + \frac{1}{3}y^3 = z$, and $2\dot{x} + \frac{1}{3}\dot{x}^{\frac{3}{2}} = \dot{z}$.

DEMONSTRATION.

56. And thus we have solved the Problem, but the Demonstration is still behind. And in so great a variety of matters, that we may not derive it synthetically, and with too great perplexity, from its genuine foundations, it may be sufficient to point it out thus in short, by way of Analysis. That is, when any Equation is propos'd, after you have finish'd the work, you may try whether from the derived Equation you can return back to the Equation propos'd, by Prob. 1. And therefore, the Relation of the Quantities in the derived Equation requires the Relation of the Fluxions in the propos'd Equation, and contrary-wise: which was to be shewn.

57. So if the Equation propos'd were $y = x$, the derived Equation will be $y = \frac{1}{2}x^2$; and on the contrary, by Prob. 1. we have $y = xx$, that is, $y = x$, because x is supposed Unity. And thus
from

from $y = 1 - 3x + y + xx + xy$ is derived $y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c. And thence by Prob. I. $y = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5$, &c. Which two Values of y agree with each other, as appears by substituting $x - xx + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5$, &c. instead of y in the first Value.

53. But in the Reduction of Equations I made use of an Operation, of which also it will be convenient to give some account. And that is, the Transmutation of a flowing Quantity by its connexion with a given Quantity. Let AE and ae be two Lines indefinitely extended each way, along which two moving Things or Points may pass from afar, and at the same time may reach the places A and a , B and b , C and c , D and d , &c. and let B be the Point, by its distance from which, the Motion of the moving thing or point in AE is estimated; so that — BA, BC, BD, BE, successively, may be the flowing Quantities, when the moving thing is in the places A, C, D, E. Likewise let b be a like point in the other Line. Then will — BA and — ba be contemporaneous Fluents, as also BC and bc , BD and bd , BE and be , &c. Now if instead of the points B and b , be substituted A and a , to which, as at rest, the Motions are refer'd; then o and — ca , AB and — cb , AC and o, AD and cd , AE and ce , will be contemporaneous flowing Quantities. Therefore the flowing Quantities are changed by the Addition and Subtraction of the given Quantities AB and ac ; but they are not changed as to the Celerity of their Motions, and the mutual respect of their Fluxion. For the contemporaneous parts AB and ab , BC and bc , CD and cd , DE and de , are of the same length in both cases. And thus in Equations in which these Quantities are represented, the contemporaneous parts of Quantities are not therefore changed, notwithstanding their absolute magnitude may be increased or diminished by some given Quantity. Hence the thing proposed is manifest: For the only Scope of this Problem is, to determine the contemporaneous Parts, or the contemporary Differences of the absolute Quantities v , x , y , or z , described with a given Rate of Flowing. And it is all one of what absolute magnitude those Quantities are, so that their contemporary or correspondent Differences may agree with the proposed Relation of the Fluxions.

59. The reason of this matter may also be thus explain'd Algebraically. Let the Equation $y = xxy$ be proposed, and suppose

pose $x = 1 + z$. Then by Prob. 1. $\dot{x} = \dot{z}$. So that for $y = xxy$, may be wrote $y = xy + xzy$. Now since $x = z$, it is plain, that though the Quantities x and z be not of the same length, yet that they flow alike in respect of y , and that they have equal contemporaneous parts. Why therefore may I not represent by the same Symbols Quantities that agree in their Rate of Flowing; and to determine their contemporaneous Differences, why may not I use $y = xy + xxy$ instead of $y = xxy$?

60. Lastly it appears plainly in what manner the contemporary parts may be found, from an Equation involving flowing Quantities. Thus if $y = \frac{1}{x} + x$ be the Equation, when $x = 2$, then $y = 2\frac{1}{2}$. But when $x = 3$, then $y = 3\frac{1}{3}$. Therefore while x flows from 2 to 3, y will flow from $2\frac{1}{2}$ to $3\frac{1}{3}$. So that the parts described in this time are $3 - 2 = 1$, and $3\frac{1}{3} - 2\frac{1}{2} = \frac{5}{6}$.

61. This Foundation being thus laid for what follows, I shall now proceed to more particular Problems.

P R O B. III.

To determine the Maxima and Minima of Quantities.

1. When a Quantity is the greatest or the least that it can be, at that moment it neither flows backwards or forwards. For if it flows forwards, or increases, that proves it was less, and will presently be greater than it is. And the contrary if it flows backwards, or decreases. Wherefore find its Fluxion, by Prob. 1. and suppose it to be nothing.

2. EXAMP. 1. If in the Equation $x^3 - ax^2 + axy - y^3 = 0$ the greatest Value of x be required; find the Relation of the Fluxions of x and y , and you will have $3xx^2 - 2axx + axy - 3yy^2 + ayx = 0$. Then making $\dot{x} = 0$, there will remain $-3yy^2 + ayx = 0$, or $3y^2 = ax$. By the help of this you may exterminate either x or y out of the primary Equation, and by the resulting Equation you may determine the other, and then both of them by $-3y^2 + ax = 0$.

3. This Operation is the same, as if you had multiply'd the Terms of the proposed Equation by the number of the Dimensions of the other flowing Quantity y . From whence we may derive the famous

famous Rule of *Huddenius*, that, in order to obtain the greatest or least Relate Quantity, the Equation must be disposed according to the Dimensions of the Correlate Quantity, and then the Terms are to be multiply'd by any Arithmetical Progression. But since neither this Rule, nor any other that I know yet published, extends to Equations affected with surd Quantities, without a previous Reduction; I shall give the following Example for that purpose.

4. EXAMP. 2. If the greatest Quantity y in the Equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ be to be determin'd, seek the Fluxions of x and y , and there will arise the Equation $3xx^2 - 2ayy + \frac{3aby^2 + 2byy^3}{a^2 + 2ay + y^2} - \frac{4axxy + 6xx^2 + ayx^2}{2\sqrt{ay+xx}} = 0$. And since by supposition $\dot{y} = 0$, omit the Terms multiply'd by y , (which, to shorten the labour, might have been done before, in the Operation,) and divide the rest by xx , and there will remain $3x - \frac{2ay + 3xx}{\sqrt{ay+xx}} = 0$. When the Reduction is made, there will arise $4ay + 3xx = 0$, by help of which you may exterminate either of the quantities x or y out of the propos'd Equation, and then from the resulting Equation, which will be Cubical, you may extract the Value of the other.

5. From this Problem may be had the Solution of these following.

I. In a given Triangle, or in a Segment of any given Curve, to inscribe the greatest Rectangle.

II. To draw the greatest or the least right Line, which can lie between a given Point, and a Curve given in position. Or, to draw a Perpendicular to a Curve from a given Point.

III. To draw the greatest or the least right Lines, which passing through a given Point, can lie between two others, either right Lines or Curves.

IV. From a given Point within a Parabola, to draw a right Line, which shall cut the Parabola more obliquely than any other. And to do the same in other Curves.

V. To determine the Vertices of Curves, their greatest or least Breadths, the Points in which revolving parts cut each other, &c.

VI. To find the Points in Curves, where they have the greatest or least Curvature.

VII. To find the least Angle in a given Ellipsis, in which the Ordinates can cut their Diameters.

VIII. Of Ellipses that pass through four given Points, to determine the greatest, or that which approaches nearest to a Circle.

IX. To determine such a part of a Spherical Superficies, which can be illuminated, in its farther part, by Light coming from a great distance, and which is refracted by the nearer Hemisphere.

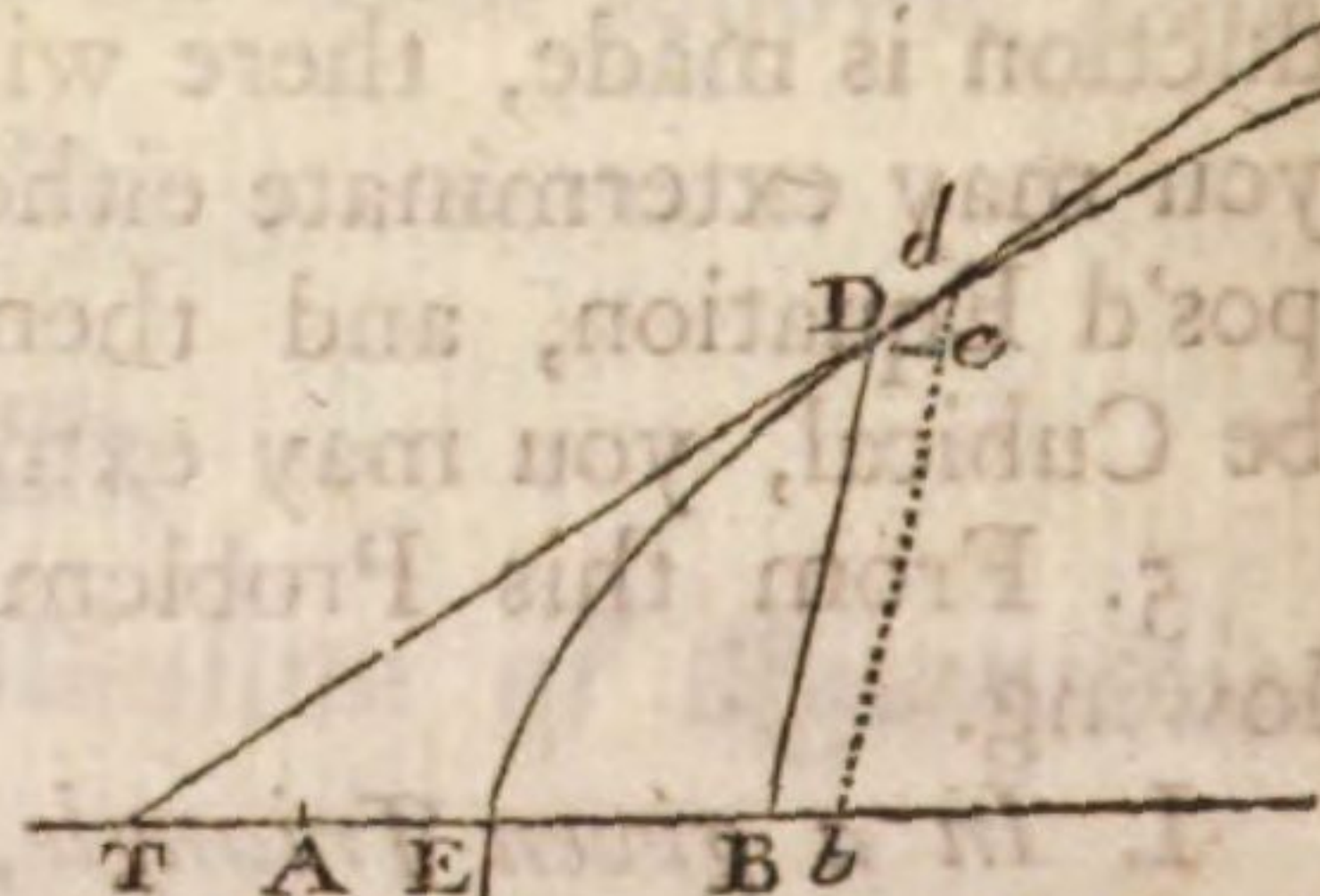
And many other Problems of a like nature may more easily be proposed than resolved, because of the labour of Computation.

P R O B. IV.

To draw Tangents to Curves.

First Manner.

1. Tangents may be variously drawn, according to the various Relations of Curves to right Lines. And first let BD be a right Line, or Ordinate, in a given Angle to another right Line AB, as a Base or Absciss, and terminated at the Curve ED. Let this Ordinate move through an indefinitely small Space to the place *bd*, so that it may be increased by the Moment *cd*, while AB is increased by the Moment *Bb*, to which *Dc* is equal and parallel. Let *Dd* be produced till it meets with AB in T, and this Line will touch the Curve in D or *d*; and the Triangles *dcD*, DBT will be similar. So that it is $TB : BD :: Dc \text{ (or } Bb) : cd$.



2. Since therefore the Relation of BD to AB is exhibited by the Equation, by which the nature of the Curve is determined; seek for the Relation of the Fluxions, by Prob. 1. Then take TB to BD in the Ratio of the Fluxion of AB to the Fluxion of BD, and TD will touch the Curve in the Point D.

3. Ex. 1. Calling $AB = x$, and $BD = y$, let their Relation be $x^3 - ax^2 + axy - y^3 = 0$. And the Relation of the Fluxions will be $3xx^2 - 2axx + axy - 3yy^2 + ayx = 0$. So that $y : x :: 3xx - 2ax + ay : 3y^2 - ax :: BD (y) : BT$. Therefore $BT = \frac{3y^3 - axy}{3x^2 - 2ax + ay}$. Therefore the Point D being given, and thence DB and AB, or y and x , the length BT will be given, by which the Tangent TD is determined.

4. But this Method of Operation may be thus concinnated. Make the Terms of the proposed Equation equal to nothing: multiply by the proper number of the Dimensions of the Ordinate, and put the Result in the Numerator: Then multiply the Terms of the same Equation by the proper number of the Dimensions of the Absciss, and put the Product divided by the Absciss, in the Denominator of the Value of BT. Then take BT towards A, if its Value be affirmative, but the contrary way if that Value be negative.

5. Thus the Equation $x^3 - ax^2 + axy - y^3 = 0$, being multiply'd by the upper Numbers, gives $axy - 3y^3$ for the Numerator; and multiply'd by the lower Numbers, and then divided by x , gives $3x^2 - 2ax + ay$ for the Denominator of the Value of BT.

6. Thus the Equation $y^3 - by^2 - cdy + bcd + dxy = 0$, (which denotes a Parabola of the second kind, by help of which *Des Cartes* constructed Equations of six Dimensions; see his *Geometry*, p. 42. *Amsterd. Ed. An. 1659.*) by Inspection gives $\frac{3y^3 - 2by^2 - cdy + dxy}{dy}$, or $\frac{3yy}{d} - \frac{2by}{d} - c + x = \text{BT}$.

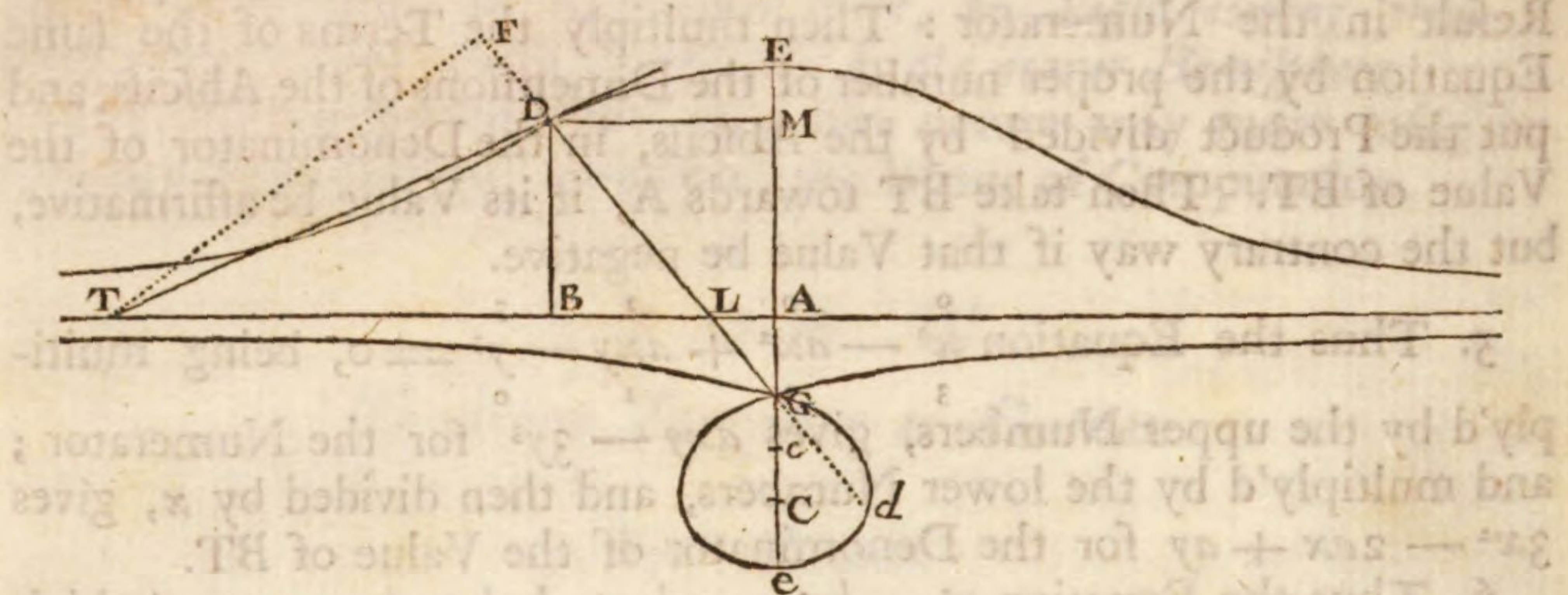
7. And thus $a^2 - \frac{r}{q}x^2 - y^2 = 0$, (which denotes an Ellipsis whose Center is A,) gives $\frac{-\frac{2yy}{2r}}{-\frac{x}{q}}$, or $\frac{qyy}{rx} = \text{BT}$. And so in others.

8. And you may take notice, that it matters not of what quantity the Angle of Ordination ABD may be.

9. But as this Rule does not extend to Equations affected by surd Quantities, or to mechanical Curves; in these Cases we must have recourse to the fundamental Method.

10. Ex. 2. Let $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ be the Equation expressing the Relation between AB and BD; and by Prob. 1. the Relation of the Fluxions will be $3xx^2 - 2ayy + \frac{3aby^2 + 2byy^3}{aa + 2ay + yy} - \frac{4axxy - 6xx^2 - ayx^2}{2\sqrt{ay+xx}} = 0$. Therefore it will be $3xx \frac{4axy - 6x^3}{2\sqrt{ay+xx}} : 2ay \frac{-3aby + 2by^3}{aa + 2ay + yy} + \frac{axx}{2\sqrt{ay+xx}} :: (\dot{y} : \dot{x} ::) \text{BD} : \text{BT}$.

11. Ex. 3. Let ED be the Conchoid of *Nicomedes*, described with the Pole G, the Asymptote AT, and the Distance LD; and let



$GA = b$, $LD = c$, $AB = x$, and $BD = y$. And because of similar Triangles DBL and DMG, it will be $LB : BD :: DM : MG$; that is, $\sqrt{cc - yy} : y :: x : b + y$, and therefore $b + y \sqrt{cc - yy} = yx$. Having got this Equation, I suppose $\sqrt{cc - yy} = z$, and thus I shall have two Equations $bz + yz = yx$, and $zz = cc - yy$. By the help of these I find the Fluxions of the Quantities x , y , and z , by Prob. 1. From the first arises $b\dot{z} + y\dot{z} + y\dot{x} = \dot{y}x + xy\dot{x}$, and from the second $2z\dot{z} = -2y\dot{y}$, or $\dot{z}z + y\dot{y} = 0$. Out of these if we exterminate $\dot{z}$, there will arise $-\frac{by\dot{y}}{z} - \frac{y\dot{y}^2}{z} + y\dot{x} = \dot{y}x + xy\dot{x}$, which being resolved it will be $y : z - \frac{by}{z} - \frac{y\dot{y}}{z} - x :: (\dot{y} : \dot{x} ::) BD : BT$. But as BD is y , therefore $BT = z - x - \frac{by - yy}{z}$. That is, $-BT = AL + \frac{BD \times GM}{BL}$; where the Sign $-$ prefixt to BT denotes, that the Point T must be taken contrary to the Point A .

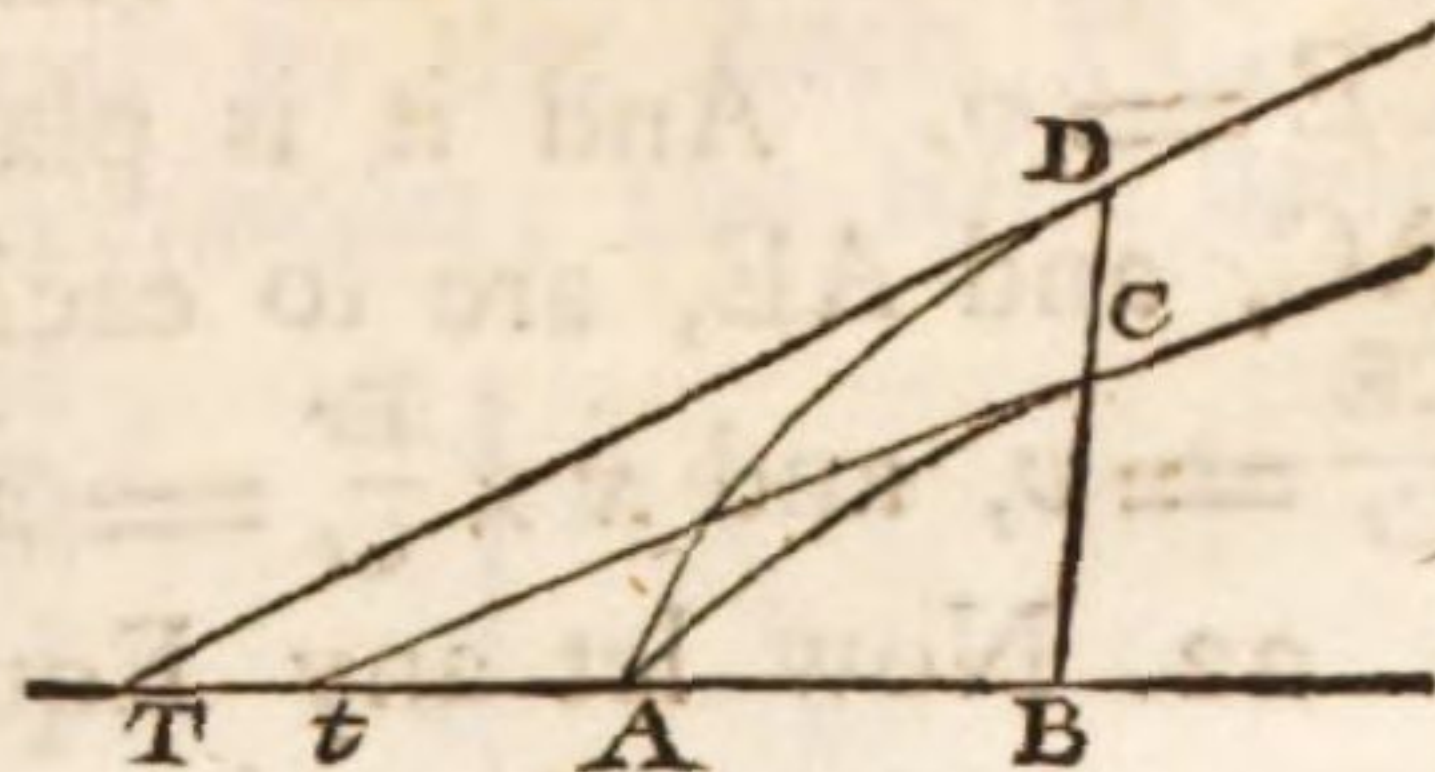
12. SCHOLIUM. And hence it appears by the bye, how that point of the Conchoid may be found, which separates the concave from the convex part. For when AT is the least possible, D will be that point. Therefore make $AT = v$; and since $BT = -z + x + \frac{by + yy}{z}$, then $v = -z + 2x + \frac{by + yy}{z}$. Here to shorten the work, for x substitute $\frac{bz + yz}{y}$, which Value is derived from what is before, and it will be $\frac{2bz}{y} + z + \frac{by + yy}{z} = v$. Whence the Fluxions $\dot{v}$, $\dot{y}$, and $\dot{z}$ being found by Prob. 1. and supposing $\dot{v} = 0$,
by

by Prob. 3. there will arise $\frac{2bz}{y} - \frac{2byz}{yy} + \dot{z} + \frac{by + zyy}{z} - \frac{bzy + zyy}{zz} = \dot{v} = 0$.

Lastly, substituting in this $\frac{-yy}{z}$ for $\dot{z}$, and $cc - yy$ for zz , (which values of z and zz are had from what goes before,) and making a due Reduction, you will have $y^3 + 3by^2 - 2bc^2 = 0$. By the Construction of which Equation y or AM, will be given. Then thro' M drawing MD parallel to AB, it will fall upon the Point D of contrary Flexure.

13. Now if the Curve be Mechanical whose Tangent is to be drawn, the Fluxions of the Quantities are to be found, as in Examp. 5. of Prob. 1. and then the rest is to be perform'd as before.

14. Ex. 4. Let AC and AD be two Curves, which are cut in the Points C and D by the right Line BCD, apply'd to the Absciss AB in a given Angle. Let $AB = x$, $BD = y$, and $\frac{\text{Area ACB}}{1} = z$. Then (by Prob. 1. Preparat. to Examp. 5.) it will be $\dot{z} = \dot{x} \times BC$.



15. Now let AC be a Circle, or any known Curve; and to determine the other Curve AD, let any Equation be proposed, in which z is involved, as $zz + axz = y^4$. Then by Prob. 1. $2zz + axz + ax\dot{z} = 4yy^3$. And writing $\dot{x} \times BC$ for $\dot{z}$, it will be $2xz \times BC + axx \times BC + axz = 4yy^3$. Therefore $2z \times BC + ax \times BC + az : 4y^3 :: (y : \dot{x} ::) BD : BT$. So that if the nature of the Curve AC be given, the Ordinate BC, and the Area ACB or z ; the Point T will be given, through which the Tangent DT will pass.

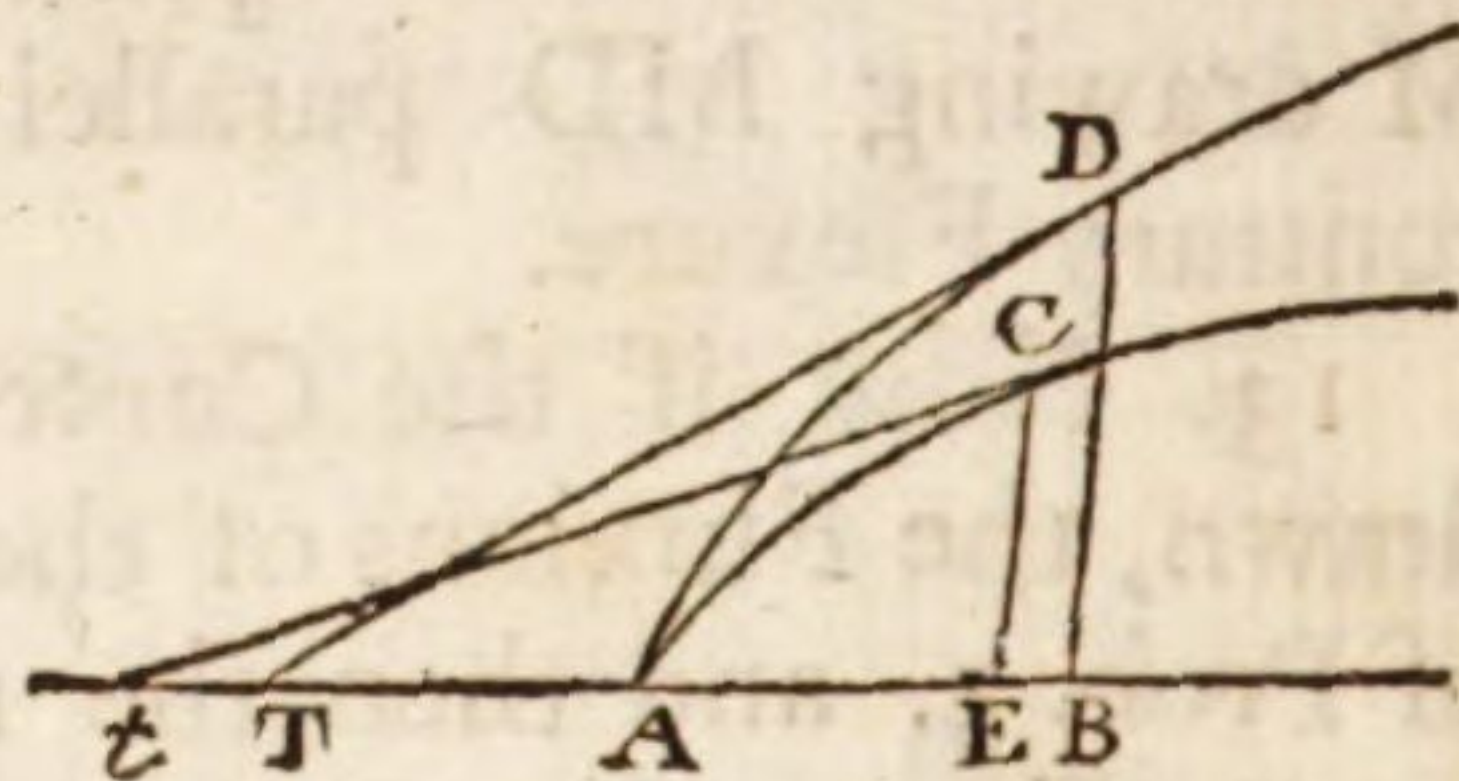
16. After the same manner, if $3z = 2y$ be the Equation to the Curve AD; 'twill be $(3z) 3\dot{x} \times BC = 2y$. So that $3BC : 2 :: (y : \dot{x} ::) BD : BT$. And so in others.

17. Ex. 5. Let $AB = x$, $BD = y$, as before, and let the length of any Curve AC be z . And drawing a Tangent to it, as Ct, 'twill be $Bt : Ct :: \dot{x} : \dot{z}$, or $\dot{z} = \frac{\dot{x} \times Ct}{Bt}$.

18. Now for determining the other Curve AD, whose Tangent is to be drawn, let there be given any Equation in which z is involved, suppose $z = y$. Then it will be $\dot{z} = \dot{y}$, so that $Ct : Bt :: (y : \dot{x} ::) : BD : BT$. But the Point T being found, the Tangent DT may be drawn.

19. Thus supposing $xz = yy$, 'twill be $\dot{x}z + z\dot{x} = 2y\dot{y}$; and for z writing $\frac{\dot{x} \times Ct}{Bt}$, there will arise $\dot{x}z + \frac{\dot{x} \times Ct}{Bt} = 2y\dot{y}$. Therefore $z + \frac{x \times Ct}{Bt} : 2y :: BD : DT$.

20. Ex. 6. Let AC be a Circle, or any other known Curve, whose Tangent is Ct, and let AD be any other Curve whose Tangent DT is to be drawn, and let it be defin'd by assuming $AB =$ to the Arch AC; and (CE, BD being Ordinates to AB in a given Angle,) let the Relation of BD to CE or AE be express'd by any Equation.



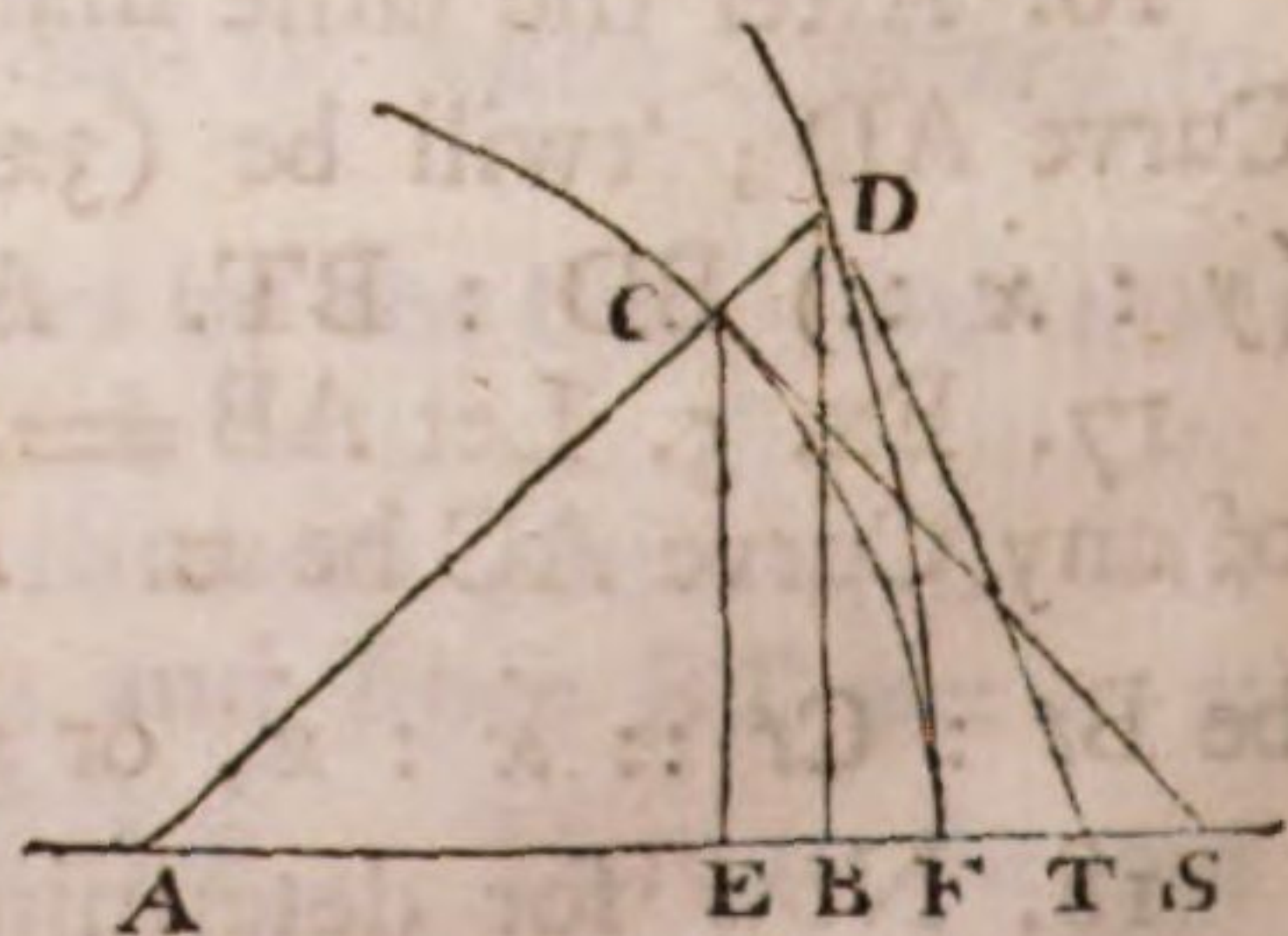
21. Therefore call AB or $AC = x$, $BD = y$, $AE = z$, and $CE = v$. And it is plain that $\dot{v}$, $\dot{x}$, and $\dot{z}$, the Fluxions of CE , AC , and AE , are to each other as CE , Ct , and Et . Therefore $\dot{x} \times \frac{CE}{Ct} = \dot{v}$, and $\dot{x} \times \frac{Et}{Ct} = \dot{z}$.

22. Now let any Equation be given to define the Curve AD, as $y = z$. Then $\dot{y} = \dot{z}$; and therefore $Et : Ct :: (\dot{y} : \dot{x} ::) BD : BT$.

23. Or let the Equation be $y = z + v - x$, and it will be $\dot{y} = (\dot{v} + \dot{z} - \dot{x}) \dot{x} \times \frac{CE + Et - Ct}{Ct}$. And therefore $CE + Et - Ct : Ct :: (\dot{y} : \dot{x} ::) BD : BT$.

24. Or finally, let the Equation be $ayy = v^3$, and it will be $2ay\dot{y} = (3v\dot{v}) = 3\dot{x}v^2 \times \frac{CE}{Ct}$. So that $3v^2 \times CE : 2ay \times Ct :: BD : BT$.

25. Ex. 7. Let FC be a Circle, which is touched by CS in C; and let FD be a Curve, which is defined by assuming any Relation of the Ordinate DB to the Arch FC, which is intercepted by DA drawn to the Center. Then letting fall CE, the Ordinate in the Circle, call AC or $AF = t$, $AB = x$, $DB = y$, $AE = z$, $CE = v$, $CF = t$; and it will be $\dot{t}z = (\dot{t} \times \frac{CE}{CS} =)$



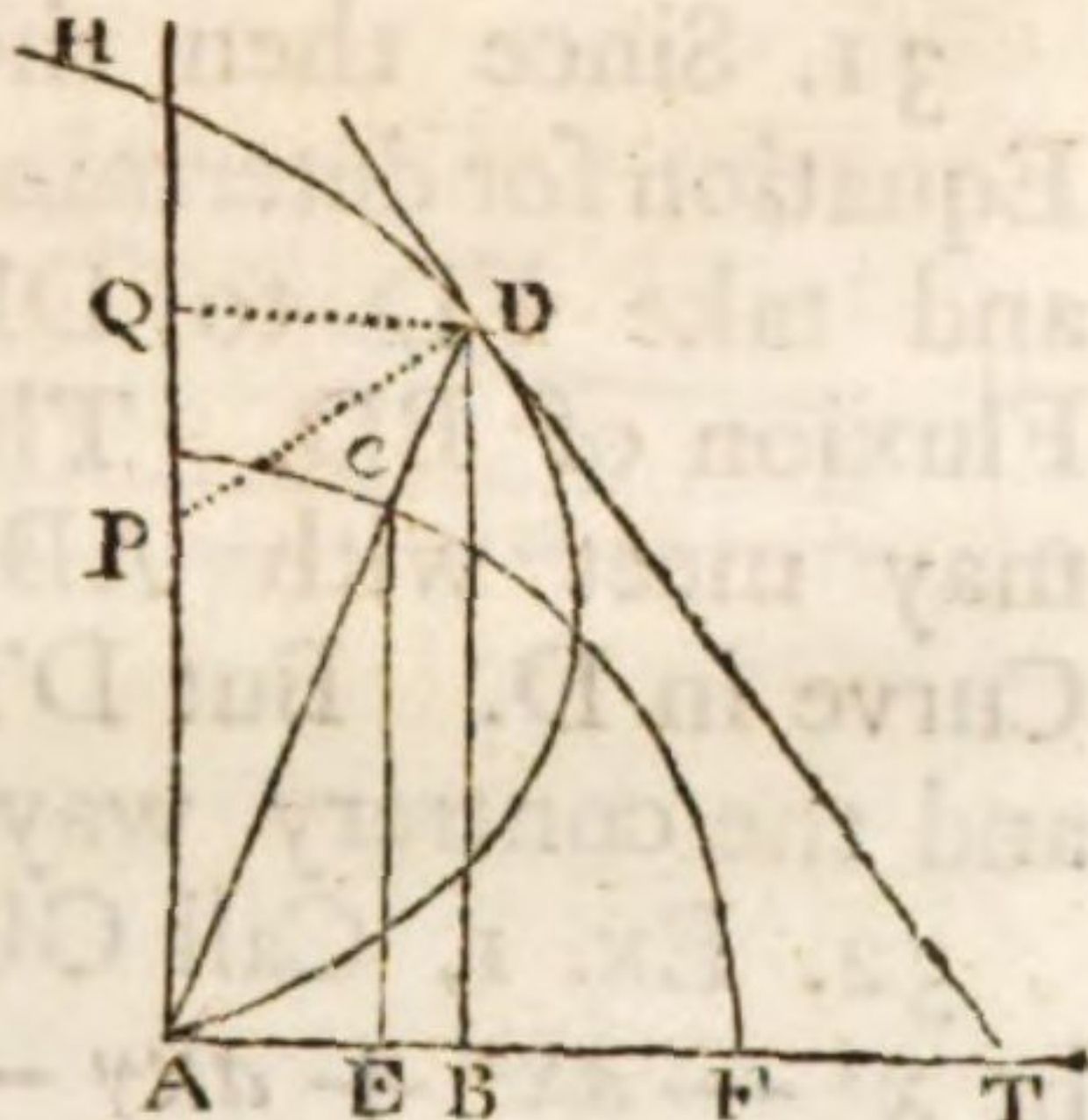
$\dot{v}$, and $-\dot{t}v = (\dot{t} \times \frac{-ES}{CS} =) \dot{z}$. Here I put $\dot{z}$ negatively, because AE is diminish'd while EC is increased. And besides $AE : EC :: AB :$

AB : BD, so that $zy = vx$, and thence by Prob. 1. $\dot{z}y + y\dot{z} = \dot{v}x + x\dot{v}$. Then exterminating $\dot{v}$, $\dot{z}$, and v , 'tis $y\dot{x} - t\dot{y}^2 - t\dot{x}^2 = xy$.

26. Now let the Curve DF be defined by any Equation, from which the Value of t may be derived, to be substituted here. Suppose let $t = y$, (an Equation to the first Quadratrix,) and by Prob. 1. it will be $t = y$, so that $\dot{y}x - \dot{y}y^2 - \dot{y}x^2 = \dot{x}y$. Whence $y : xx + yy - x :: (y : -x ::) BD (y) : BT$. Therefore $BT = x^2 + y^2 - x$; and $AT = xx + yy = \frac{ADq}{AF}$.

27. After the same manner, if it is $tt \equiv by$, there will arise $2tt \equiv by$; and thence $AT = \frac{b}{2t} \times \frac{ADq}{AF}$. And so of others.

28. Ex. 8. Now if AD be taken equal to the Arch FC, the Curve ADH being then the Spiral of *Archimedes*; the same names of the Lines still remaining as were put afore: Because of the right Angle ABD 'tis $xx + yy = tt$, and therefore (by Prob. I.) $\dot{xx} + \dot{yy} = \dot{tt}$. 'Tis also $AD : AC :: DB : CE$, so that $tv = y$, and thence (by Prob. I.) $\dot{tv} + \dot{vt} = \dot{y}$. Lastly, the Fluxion of the Arch FC is to the Fluxion of the right Line CE, as AC to AE, or as AD to AB, that is, $\dot{t} : \dot{v} :: t : x$, and thence $\dot{tx} = \dot{vt}$. Compare the Equations now found, and you will see

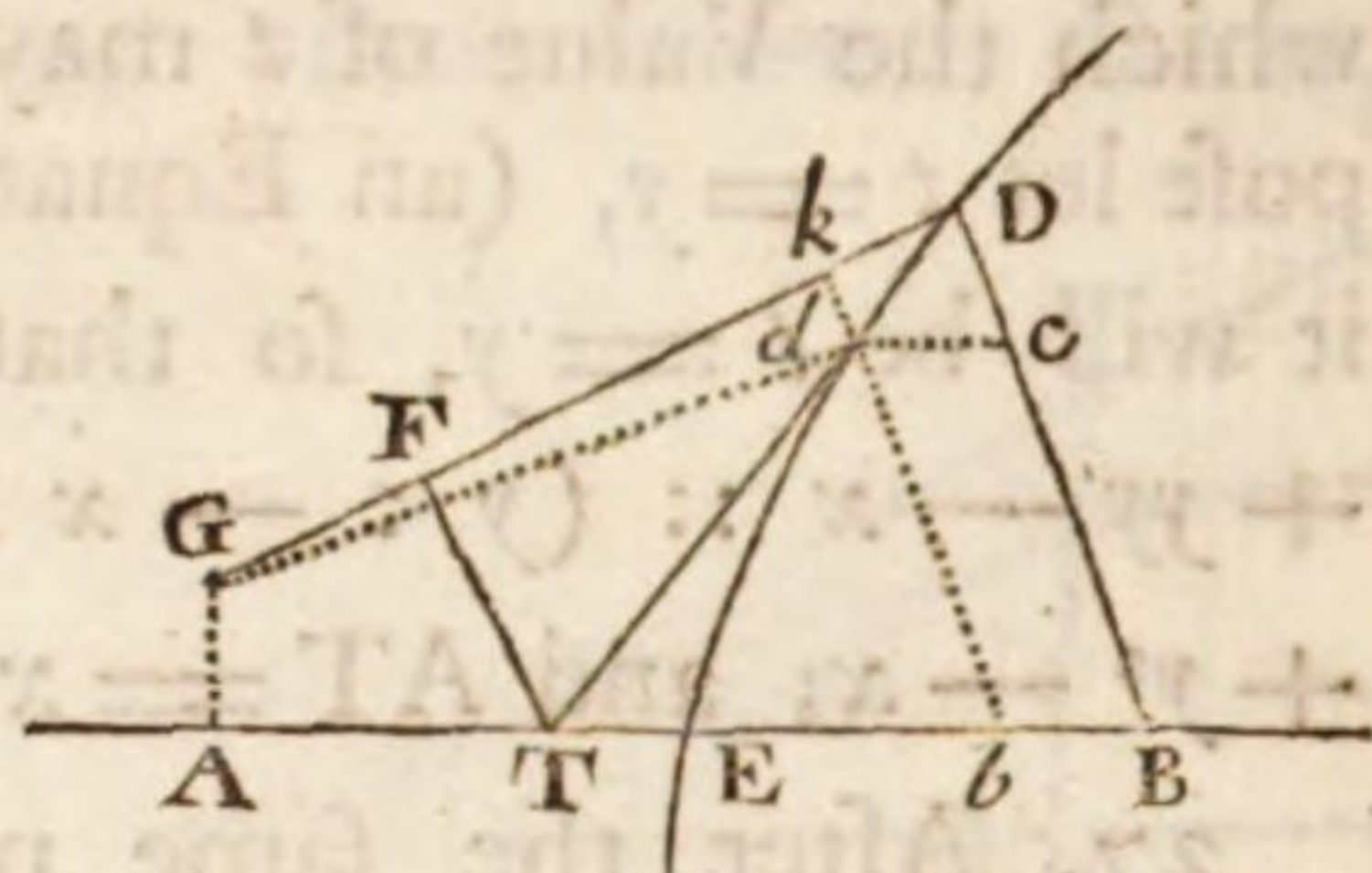


'tis $t\dot{v} + t\dot{x} = \dot{y}$, and thence $\dot{x}\dot{x} + \dot{y}\dot{y} = (\dot{t}\dot{t}) \frac{yt}{v+x}$. And therefore completing the Parallelogram ABDQ, if you make QD : QP :: (BD : BT :: $\dot{y} : -\dot{x}$::) $x : y - \frac{t}{v+x}$; that is, if you take $AP = \frac{t}{v+x}$, PD will be perpendicular to the Spiral.

29. And from hence (I imagine) it will be sufficiently manifest, by what methods the Tangents of all sorts of Curves are to be drawn. However it may not be foreign from the purpose, if I also shew how the Problem may be perform'd, when the Curves are refer'd to right Lines, after any other manner whatever : So that having the choice of several Methods, the easiest and most simple may always be used.

Second Manner.

30. Let D be a point in the Curve, from which the Subtense DG is drawn to a given Point G , and let DB be an Ordinate in any given Angle to the Absciss AB . Now let the Point D flow for an infinitely small space Dd in the Curve, and in GD let Gk be taken equal to Gd , and let the Parallelogram $dcBb$ be completed. Then Dk and Dc will be the contemporary Moments of GD and BD , by which they are diminish'd while D is transfer'd to d . Now let the right Line Dd be produced, till it meets with AB in T , and from the Point T to the Subtense GD let fall the perpendicular TF , and then the Trapezia $Dcdk$ and $DBTF$ will be like; and therefore $DB : DF :: Dc : Dk$.



31. Since then the Relation of BD to GD is exhibited by the Equation for determining the Curve; find the Relation of the Fluxions, and take FD to DB in the Ratio of the Fluxion of GD to the Fluxion of BD . Then from F raise the perpendicular FT , which may meet with AB in T , and DT being drawn will touch the Curve in D . But DT must be taken towards G , if it be affirmative, and the contrary way if negative.

32. Ex. 1. Call $GD = x$, and $BD = y$, and let their Relation be $x^3 - ax^2 + axy - y^3 = 0$. Then the Relation of the Fluxions will be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + a\dot{y}x - 3\dot{y}y^2 = 0$. Therefore $3x\dot{x} - 2ax + ay : 3yy - ax :: (y : x ::) DB (y) : DF$. So that $DF = \frac{3y^3 - axy}{3x^2 - 2ax + ay}$. Then any Point D in the Curve being given, and thence BD and GD or y and x , the Point F will be given also. From whence if the Perpendicular FT be raised, from its concurrence T with the Absciss AB , the Tangent DT may be drawn.

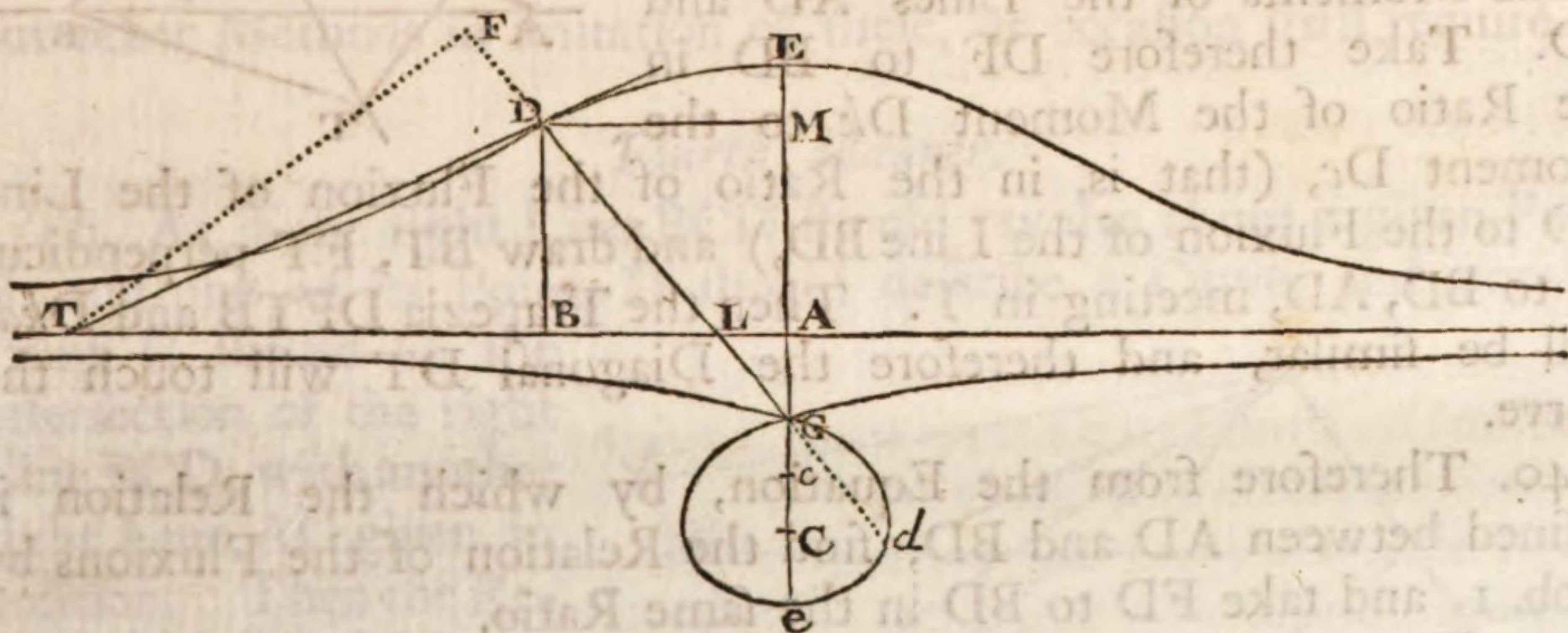
33. And hence it appears, that a Rule might be derived here, as well as in the former Case. For having disposed all the Terms of the given Equation on one side, multiply by the Dimensions of the Ordinate y , and place the result in the Numerator of a Fraction. Then multiply its Terms severally by the Dimensions of the Subtense x , and dividing the result by that Subtense x , place the Quotient in the Denominator of the Value of DF . And take the same Line DF towards G if it be affirmative, otherwise the contrary way. Where
you

you may observe, that it is no matter how far distant the Point G is from the Absciss AB, or if it be at all distant, nor what is the Angle of Ordination ABD.

34. Let the Equation be as before $x^3 - ax^2 + axy - y^3 = 0$; it gives immediately $axy - 3y^3$ for the Numerator, and $3x^2 - 2ax + ay$ for the Denominator of the Value of DF.

35. Let also $a + \frac{b}{a}x - y = 0$, (which Equation is to a Conick Section,) it gives $-y$ for the Numerator, and $\frac{b}{a}$ for the Denominator of the Value of DF, which therefore will be $-\frac{ay}{b}$.

36. And thus in the Conchoid, (wherein these things will be perform'd more expeditiously than before,) putting $GA = b$,



LD = c , GD = x , and BD = y , it will be BD (y) : DL (c) :: GA (b) : GL ($x - c$). Therefore $xy - cy = cb$, or $xy - cy - cb = 0$. This Equation according to the Rule gives $\frac{xy - cy}{y}$, that is, $x - c = DF$. Therefore prolong GD to F, so that DF = LG, and at F raise the perpendicular FT meeting the Asymptote AB in T, and DT being drawn will touch the Conchoid.

37. But when compound or surd Quantities are found in the Equation, you must have recourse to the general Method, except you should chuse rather to reduce the Equation.

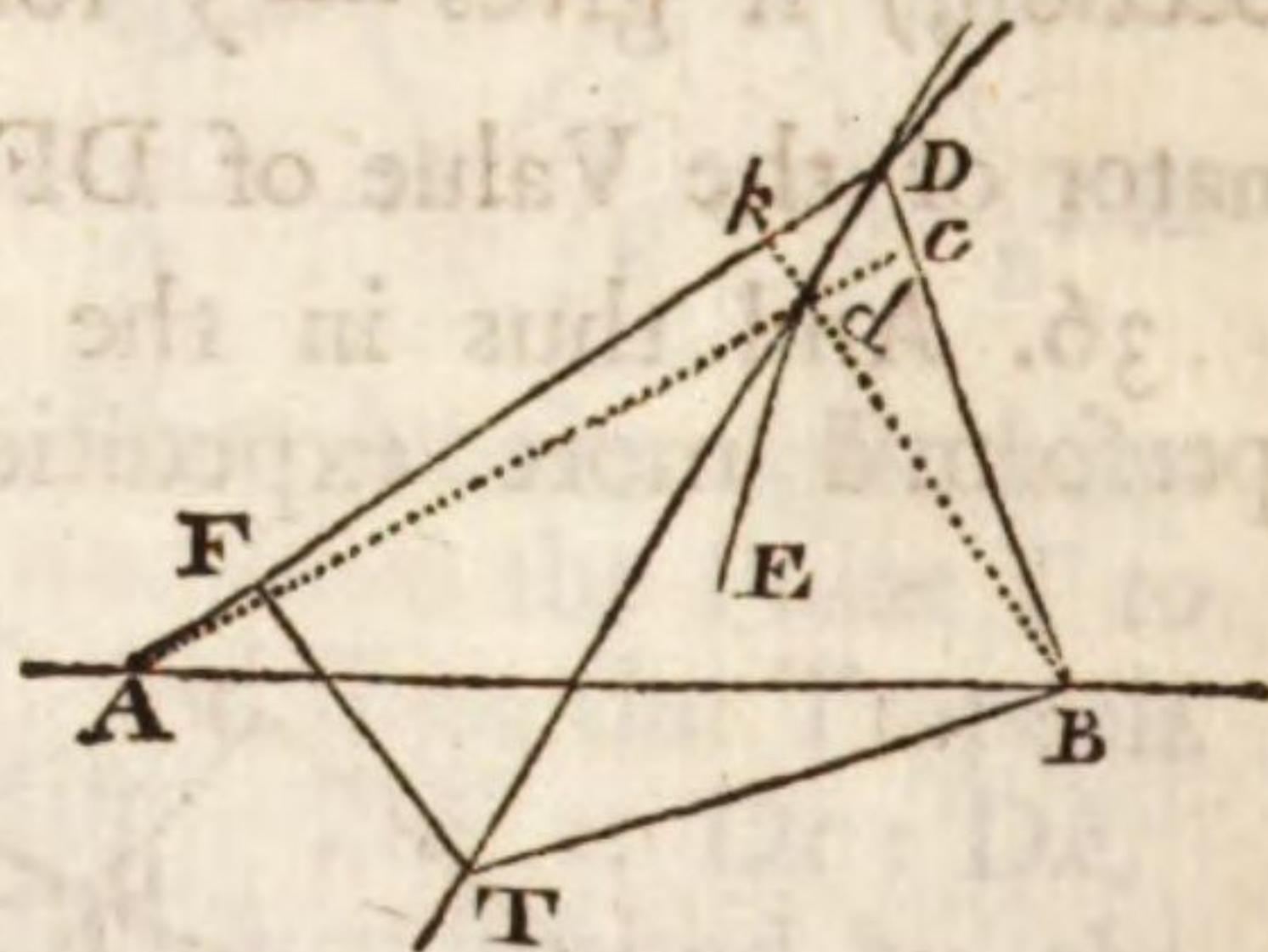
38. Ex. 2. If the Equation $\overline{b+y \times \sqrt{cc-yy}} = yx$, were given for the Relation between GD and BD; (see the foregoing Figure, p. 52.) find the Relation of the Fluxions by Prob. 1. As supposing $\sqrt{cc-yy} = z$, you will have the Equations $bz + yz = yx$, and $cc - yy = zz$, and thence the Relation of the Fluxions $bz + yz = yx + yx$, and $-2yy = 2zz$. And now z and z being exter-

exterminated, there will arise $\dot{y} \sqrt{cc - yy} - \frac{by + yy}{\sqrt{cc - yy}} - \dot{y}x = \dot{x}y$.

Therefore $y : \sqrt{cc - yy} - \frac{by + yy}{\sqrt{cc - yy}} - x :: (\dot{y} : \dot{x} ::) BD (y) : DF$.

Third Manner.

39. Moreover, if the Curve be refer'd to two Subtenses AD and BD, which being drawn from two given Points A and B, may meet at the Curve: Conceive that Point D to flow on through an infinitely little Space Dd in the Curve; and in AD and BD take Ak = Ad, and Bc = Bd; and then kD and cD will be contemporaneous Moments of the Lines AD and BD. Take therefore DF to BD in the Ratio of the Moment Dk to the Moment Dc, (that is, in the Ratio of the Fluxion of the Line AD to the Fluxion of the Line BD,) and draw BT, FT perpendicular to BD, AD, meeting in T. Then the Trapezia DFTB and Dkdc will be similar, and therefore the Diagonal DT will touch the Curve.

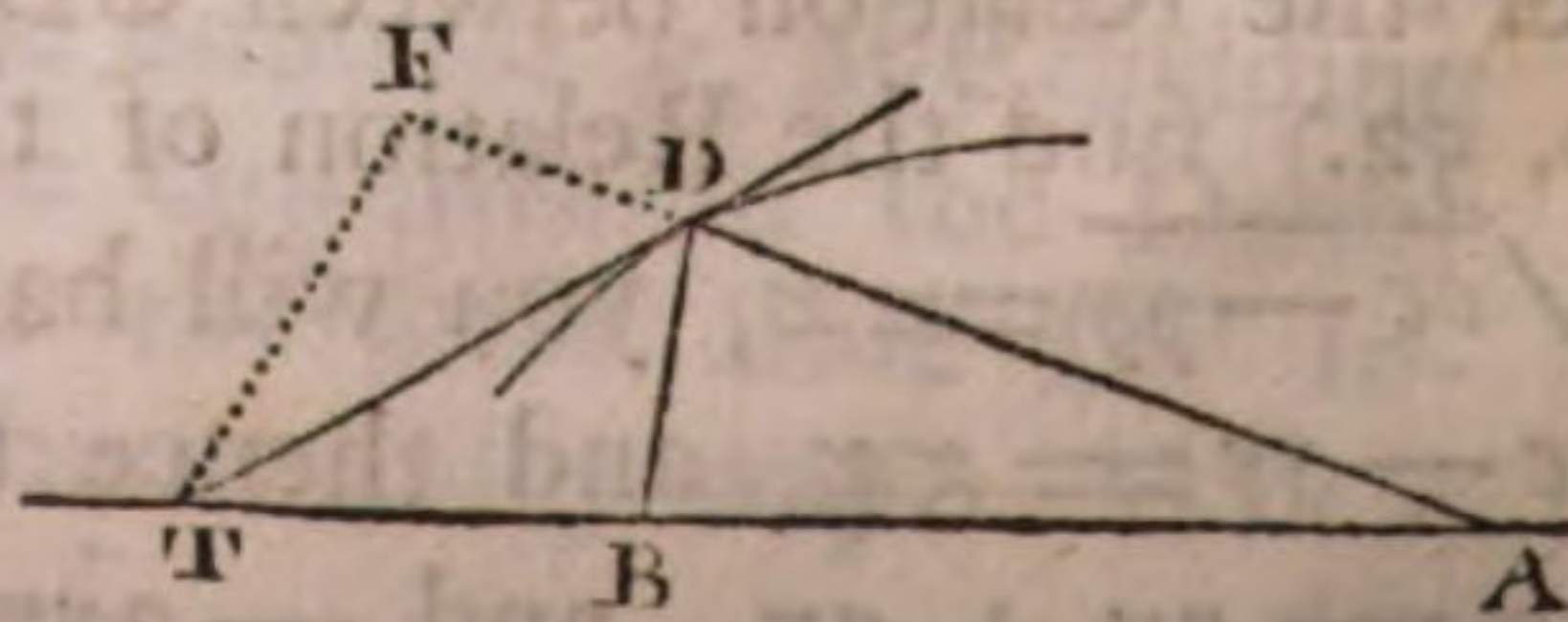


40. Therefore from the Equation, by which the Relation is defined between AD and BD, find the Relation of the Fluxions by Prob. 1. and take FD to BD in the same Ratio.

41. EXAMP. Supposing $AD = x$, and $BD = y$, let their Relation be $a + \frac{ex}{d} - y = 0$. This Equation is to the Ellipses of the second Order, whose Properties for Refracting of Light are shewn by *Des Cartes*, in the second Book of his Geometry. Then the Relation of the Fluxions will be $\frac{ex}{d} - \dot{y} = 0$. 'Tis therefore $e : d :: (\dot{y} : \dot{x} ::) BD : DF$.

42. And for the same reason if $a - \frac{ex}{d} - y = 0$, 'twill be $e : -d :: BD : DF$. In the first Case take DF towards A, and contrary-wise in the other case.

43. COROL. 1. Hence if $d = e$, (in which case the Curve becomes a Conick Section,) 'twill be $DF = DB$. And therefore the Triangles DFT and DBT being equal, the Angle FDB will be bisected by the Tangent.

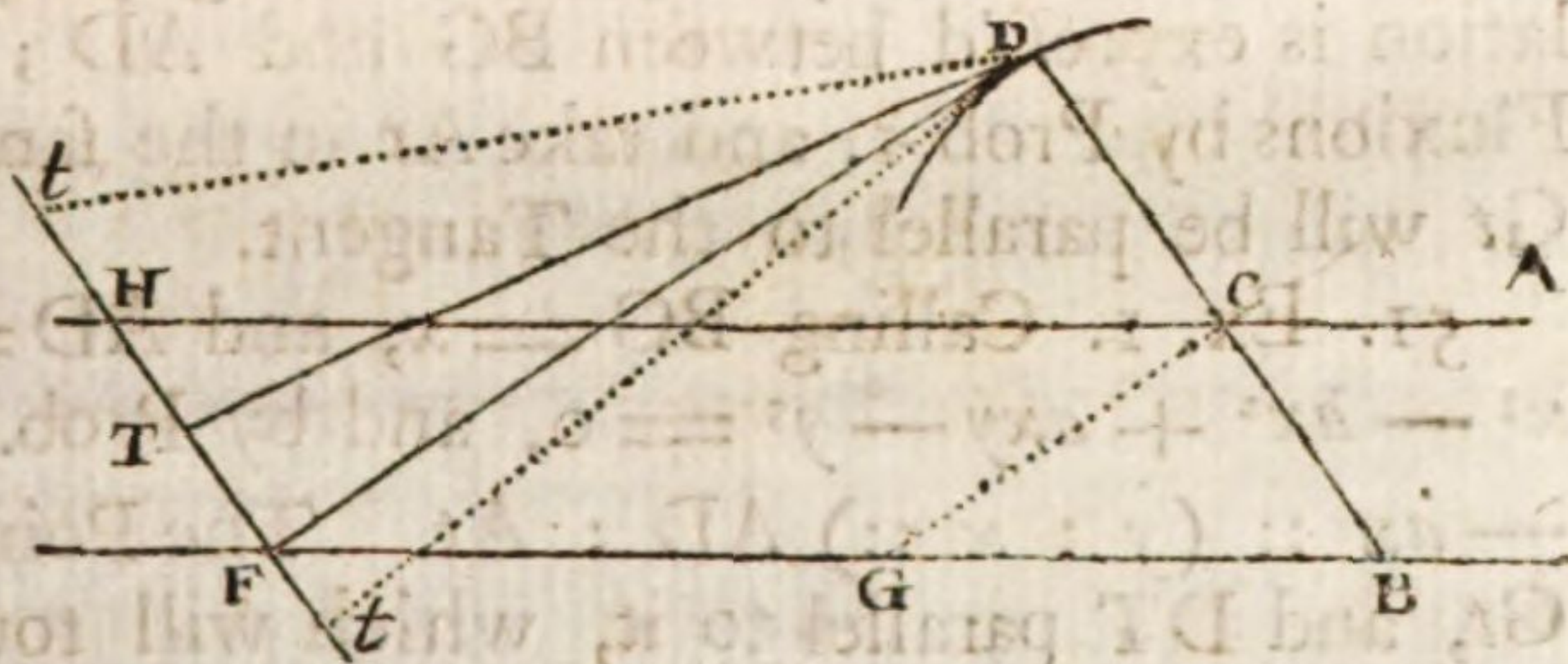


44. COROL. 2. And hence also those things will be manifest of themselves, which are demonstrated, in a very prolix manner, by *Des Cartes* concerning the Refraction of these Curves. For as much as DF and DB, (which are in the given Ratio of d to e ,) in respect of the Radius DT, are the Sines of the Angles DTF and DTB, that is, of the Ray of Incidence AD upon the Surface of the Curve, and of its Reflexion or Refraction DB. And there is a like reasoning concerning the Refractions of the Conick Sections, supposing that either of the Points A or B be conceived to be at an infinite distance.

45. It would be easy to modify this Rule in the manner of the foregoing, and to give more Examples of it: As also when Curves are refer'd to Right lines after any other manner, and cannot commodiously be reduced to the foregoing, it will be very easy to find out other Methods in imitation of these, as occasion shall require.

Fourth Manner.

46. As if the right Line BCD should revolve about a given Point B, and one of its Points D should describe a Curve, and another Point C should be the intersection of the right Line BCD, with another right Line AC given in position. Then the Relation of BC and BD being express'd by any Equation; draw BF parallel to AC, so as to meet DF, perpendicular to BD, in F. Also erect FT perpendicular to DF; and take FT in the same Ratio to BC, that the Fluxion of BD has to the Fluxion of BC. Then DT being drawn will touch the Curve.



Fifth Manner.

47. But if the Point A being given, the Equation should express the Relation between AC and BD; draw CG parallel to DF, and take FT in the same Ratio to BG, that the Fluxion of BD has to the Fluxion of AC.

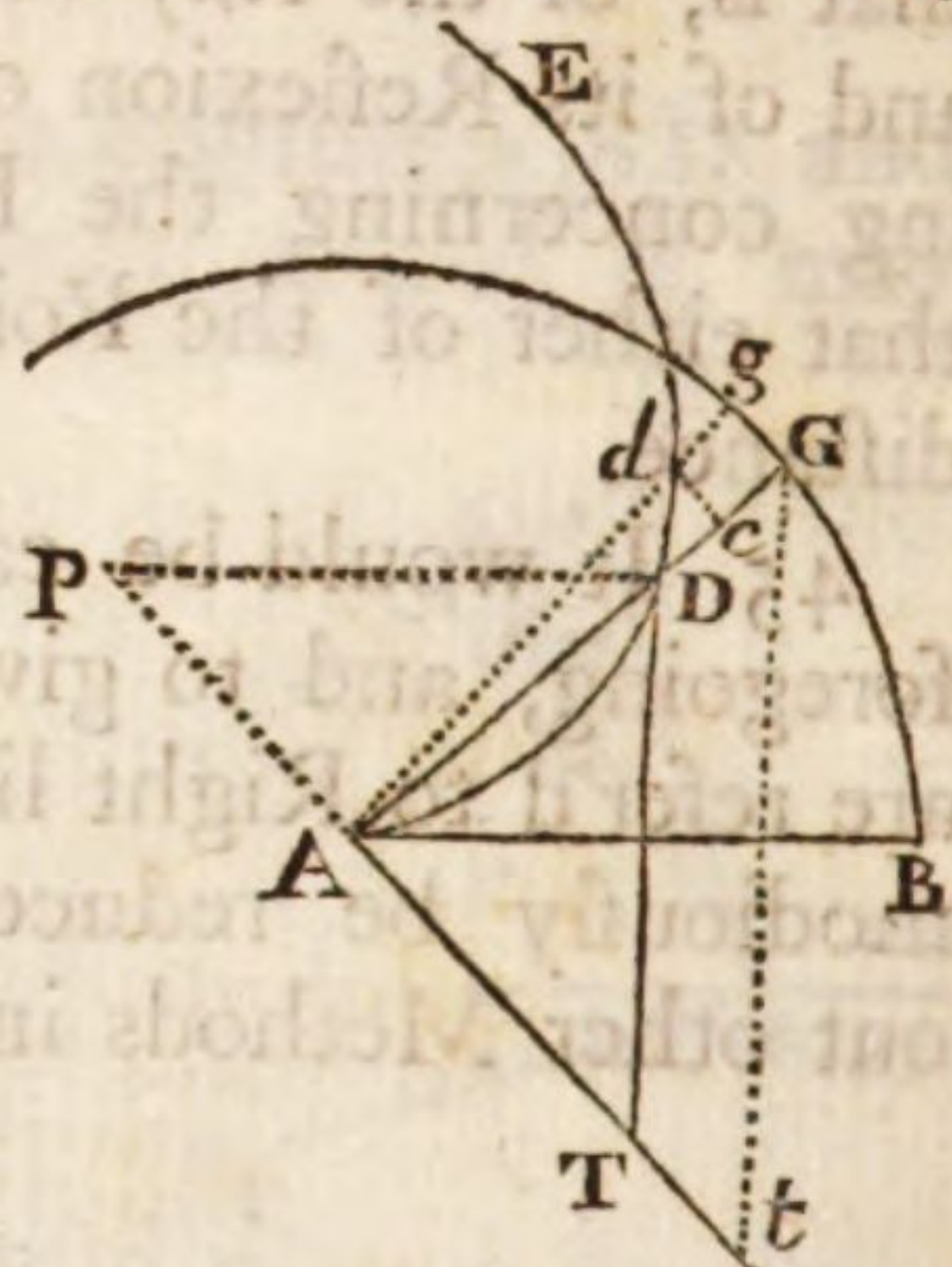
Sixth Manner.

48. Or again, if the Equation expresses the Relation between AC and CD; let AC and FT meet in H; and take HT in the same Ratio to BG, that the Fluxion of CD has to the Fluxion of AC. And the like in others. *Seventh*

Seventh

Seventh Manner : For Spirals.

49. The Problem is not otherwise perform'd, when the Curves are refer'd, not to right Lines, but to other Curve-lines, as is usual in Mechanick Curves. Let BG be the Circumference of a Circle, in whose Semidiameter AG, while it revolves about the Center A, let the Point D be conceived to move any how, so as to describe the Spiral ADE. And suppose Dd to be an infinitely little part of the Curve thro' which D flows, and in AD take $Ac = Ad$, then cD and Gg will be contemporaneous Moments of the right Line AD and of the Periphery BG. Therefore draw At parallel to cd, that is, perpendicular to AD, and let the Tangent DT meet it in T; then it will be $cD : cd :: AD : AT$. Also let Gt be parallel to the Tangent DT, and it will be $cd : Gg :: (Ad \text{ or } AD : AG ::) AT : At$.



50. Therefore any Equation being proposed, by which the Relation is express'd between BG and AD; find the Relation of their Fluxions by Prob. 1. and take At in the same Ratio to AD: And then Gt will be parallel to the Tangent.

51. Ex. 1. Calling $BG = x$, and $AD = y$, let their Relation be $x^3 - ax^2 + axy - y^3 = 0$, and by Prob. 1. $3x^2 - 2ax + ay : 3y^2 - ax :: (y : x ::) AD : At$. The Point t being thus found, draw Gt, and DT parallel to it, which will touch the Curve.

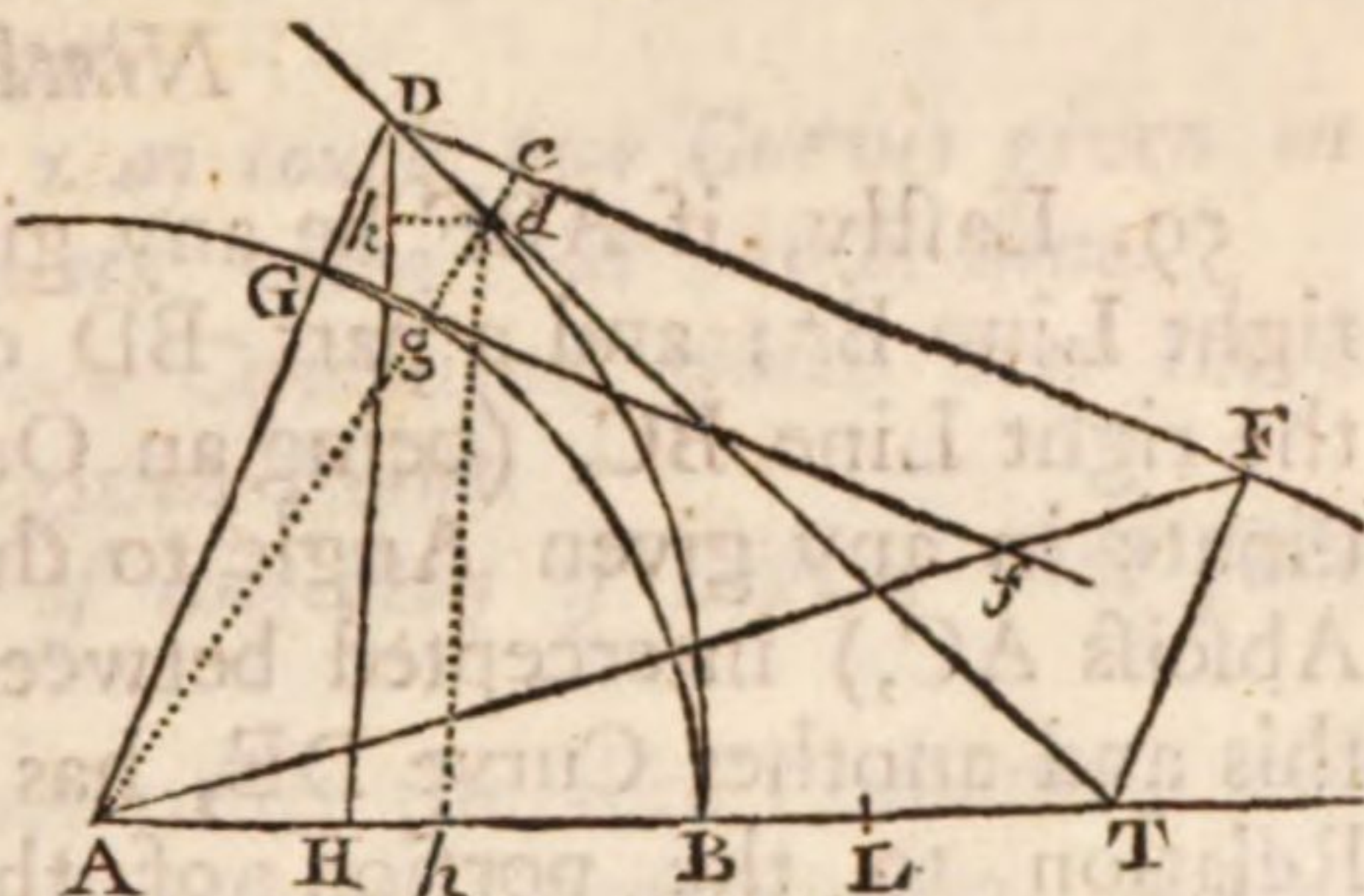
52. Ex. 2. If 'tis $\frac{ax}{b} = y$, (which is the Equation to the Spiral of Archimedes,) 'twill be $\frac{ax}{b} = \dot{y}$, and therefore $a : b :: (y : x ::) AD : At$. Wherefore by the way, if TA be produced to P, that it may be $AP : AB :: a : b$, PD will be perpendicular to the Curve.

53. Ex. 3. If $xx = by$, then $2xx = b\dot{y}$, and $2x : b :: AD : At$. And thus Tangents may be easily drawn to any Spirals what-

Eighth

Eighth Manner : For Quadratrices.

54. Now if the Curve be such, that any Line AGD, being drawn from the Center A, may meet the Circular Arch in G, and the Curve in D; and if the Relation between the Arch BG, and the right Line DH, which is an Ordinate to the Base or Absciss AH in a given Angle, be determin'd by any Equation whatever : Conceive the Point D to move in the Curve for an infinitely small Interval to d , and the Parallelogram $dbHk$ being compleated, produce Ad to c , so that $Ac = AD$; then Gg and Dk will be contemporaneous Moments of the Arch BG and of the Ordinate DH. Now produce Dd strait on to T, where it may meet with AB, and from thence let fall the Perpendicular TF on DcF . Then the Trapezia $Dkdc$ and $DHTF$ will be similar; and therefore $Dk : Dc :: DH : DF$. And besides if Gf be raised perpendicular to AG, and meets AF in f ; because of the Parallels DF and Gf , it will be $Dc : Gg :: DF : Gf$. Therefore *ex æquo*, 'tis $Dk : Gg :: DH : Gf$, that is, as the Moments or Fluxions of the Lines DH and BG.



55. Therefore by the Equation which expresses the Relation of BG to DH, find the Relation of the Fluxions (by Prob. 1.) and in that Ratio take Gf , the Tangent of the Circle BG, to DH. Draw DF parallel to Gf , which may meet Af produced in F. And at F erect the perpendicular FT, meeting AB in T; and the right Line DT being drawn, will touch the Quadratrix.

56. Ex. 1. Making $BG = x$, and $DH = y$, let it be $xx = by$; then (by Prob. 1.) $2xx = by$. Therefore $2x : b :: (y : x ::) DH : Gf$; and the Point f being found, the rest will be determin'd as above.

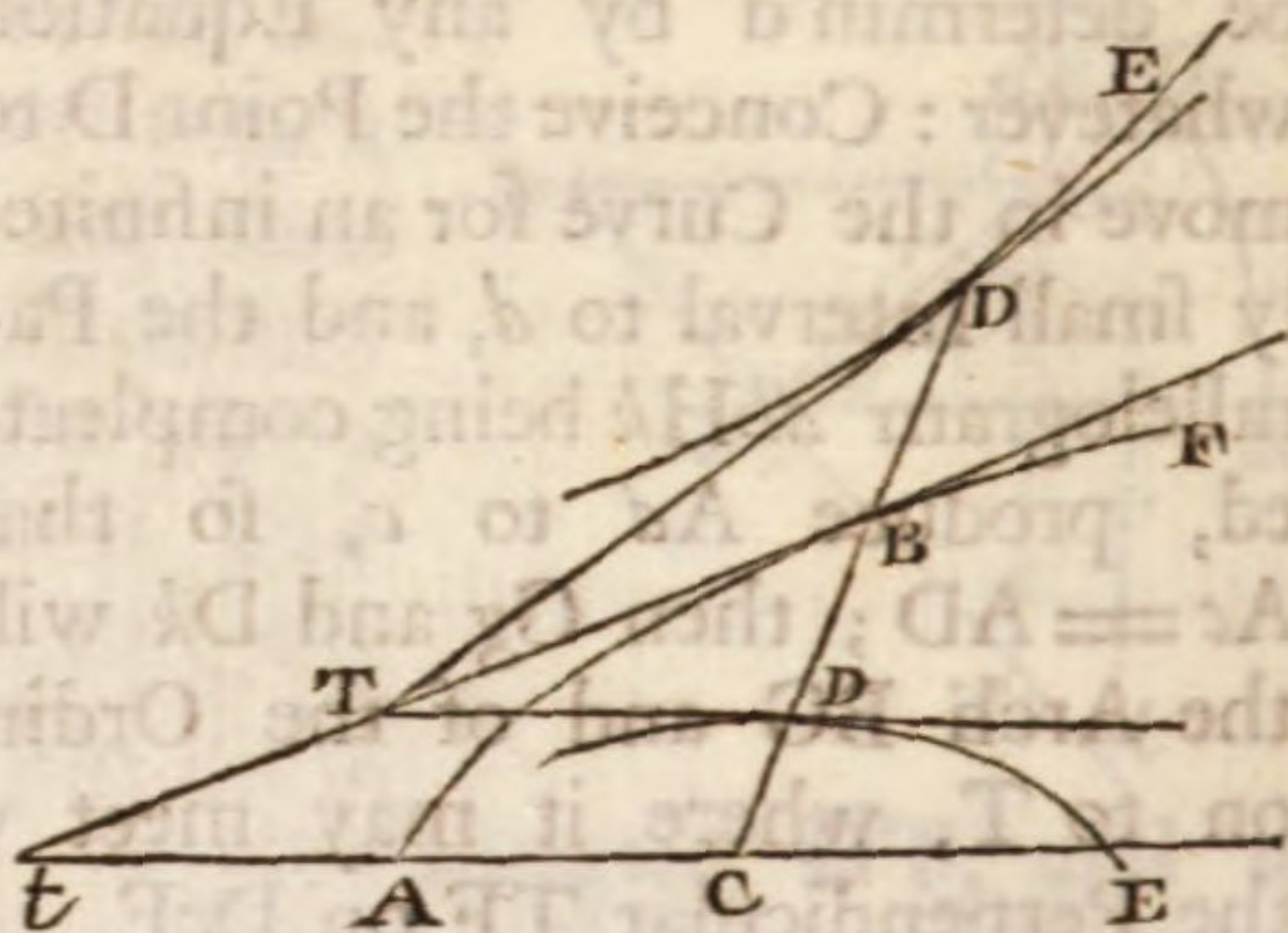
But perhaps this Rule may be thus made something neater : Make $x : y :: AB : AL$. Then $AL : AD :: AD : AT$, and then DT will touch the Curve. For because of equal Triangles AFD and ATD, 'tis $AD \times DF = AT \times DH$, and therefore $AT : AD :: (DF \text{ or } \frac{AD}{AG} \times Gf : DH \text{ or } \frac{y}{x} Gf ::) AD : (\frac{y}{x} AG \text{ or}) AL$.

57. Ex. 2. Let $x = y$, (which is the Equation to the Quadratrix of the Ancients,) then $x = y$. Therefore $AB : AD :: AD : AT$.

58. Ex. 3. Let $axx = y^3$, then $2axx = 3yy^2$. Therefore make $3y^2 : 2ax :: (\dot{x} : \dot{y} ::) AB : AL$. Then $AL : AD :: AD : AT$. And thus you may determine expeditiously the Tangents of any other Quadratrices, howsoever compounded.

Ninth Manner.

59. Lastly, if ABF be any given Curve, which is touch'd by the right Line Bt; and a part BD of the right Line BC, (being an Ordinate in any given Angle to the Absciss AC,) intercepted between this and another Curve DE, has a Relation to the portion of the Curve AB, which is express'd by any Equation: You may draw a Tangent DT to the other Curve, by taking (in the Tangent of this Curve,) BT in the same Ratio to BD, as the Fluxion of the Curve AB hath to the Fluxion of the right Line BD.



60. Ex. 1. Calling $AB = x$, and $BD = y$; let it be $ax = yy$, and therefore $ax = 2yy$. Then $a : 2y :: (\dot{y} : \dot{x} ::) BD : BT$.

61. Ex. 2. Let $\frac{a}{b}x = y$, (the Equation to the Trochoid, if ABF be a Circle,) then $\frac{a}{b}\dot{x} = \dot{y}$, and $a : b :: BD : BT$.

62. And with the same ease may Tangents be drawn, when the Relation of BD to AC, or to BC, is express'd by any Equation; or when the Curves are refer'd to right Lines, or to any other Curves, after any other manner whatever.

63. There are also many other Problems, whose Solutions are to be derived from the same Principles; such as these following.

I. To find a Point of a Curve, where the Tangent is parallel to the Absciss, or to any other right Line given in position; or is perpendicular to it, or inclined to it in any given Angle.

II. To find the Point where the Tangent is most or least inclined to the Absciss, or to any other right Line given in position. That is, to find the confine of contrary Flexure. Of this I have already given a Specimen in the Conchoid.

III. From any given Point without the Perimeter of a Curve, to draw a right Line, which with the Perimeter may make an Angle of Contact,

Contact, or a right Angle, or any other given Angle. That is, from a given Point, to draw Tangents, or Perpendiculars, or right Lines that shall have any other Inclination to a Curve-line.

IV. From any given Point within a Parabola, to draw a right Line, which may make with the Perimeter the greatest or least Angle possible. And so of all Curves whatever.

V. To draw a right Line which may touch two Curves given in position, or the same Curve in two Points, when that can be done.

VI. To draw any Curve with given Conditions, which may touch another Curve given in position, in a given Point.

VII. To determine the Refraction of any Ray of Light, that falls upon any Curve Superficies.

The Resolution of these, or of any other the like Problems, will not be so difficult, abating the tediousness of Computation, as that there is any occasion to dwell upon them here: And I imagine it may be more agreeable to Geometricians barely to have mention'd them.

P R O B. V.

At any given Point of a given Curve, to find the Quantity of Curvature.

1. There are few Problems concerning Curves more elegant than this, or that give a greater Insight into their nature. In order to its Resolution, I must premise these following general Considerations.

2. I. The same Circle has every where the same Curvature, and in different Circles it is reciprocally proportional to their Diameters. If the Diameter of any Circle is as little again as the Diameter of another, the Curvature of its Periphery will be as great again. If the Diameter be one-third of the other, the Curvature will be thrice as much, &c.

3. II. If a Circle touches any Curve on its concave side, in any given Point, and if it be of such magnitude, that no other tangent Circle can be interscribed in the Angles of Contact near that Point; that Circle will be of the same Curvature as the Curve is of, in that Point of Contact. For the Circle that comes between the Curve and another Circle at the Point of Contact, varies less from the Curve, and makes a nearer approach to its Curvature, than that other Circle does. And therefore that Circle approaches nearest to its

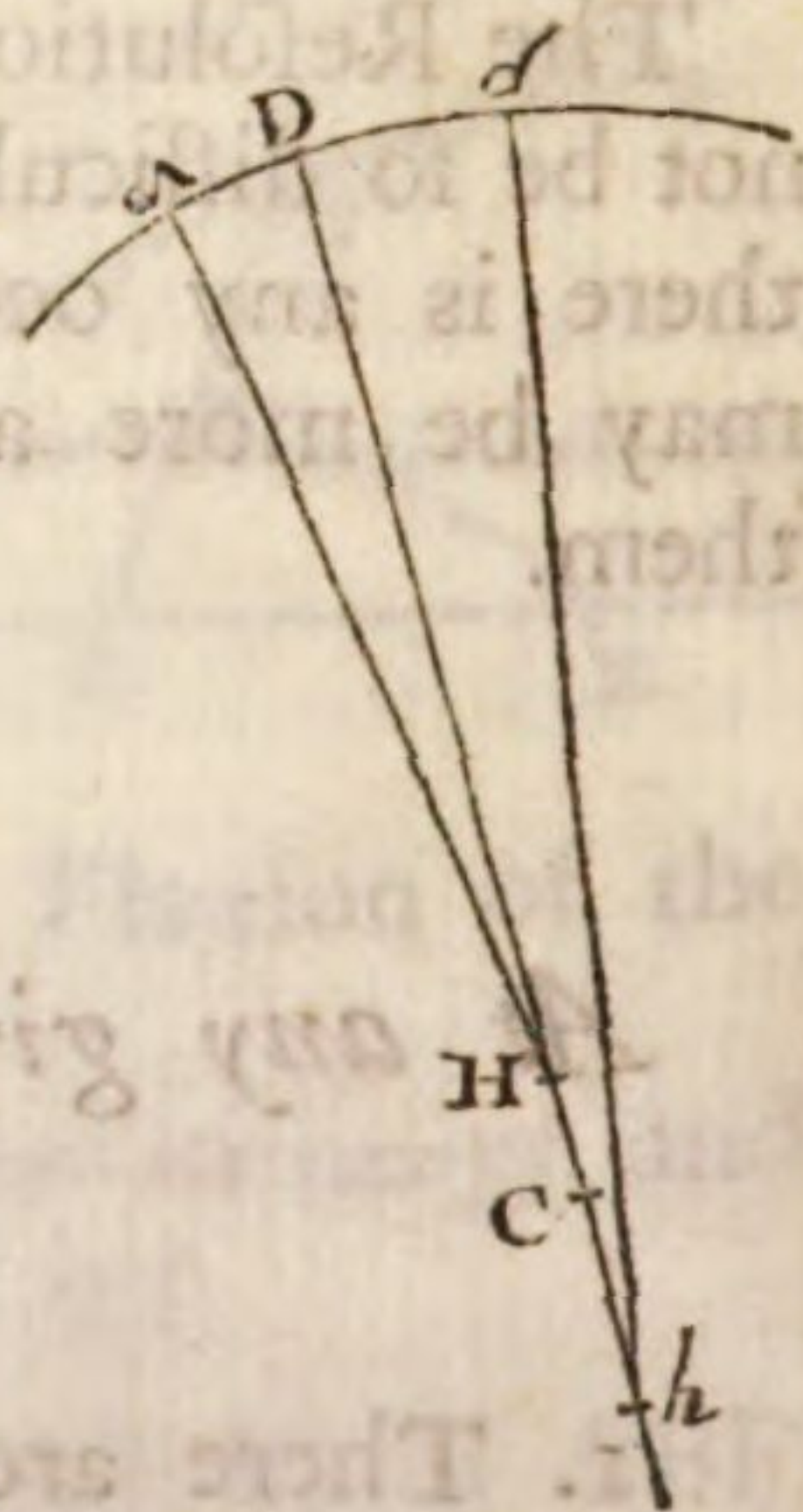
Curvature, between which and the Curve no other Circle can intervene.

4. III. Therefore the Center of Curvature to any Point of a Curve, is the Center of a Circle equally curved. And thus the Radius or Semidiameter of Curvature is part of the Perpendicular to the Curve, which is terminated at that Center.

5. IV. And the proportion of Curvature at different Points will be known from the proportion of Curvature of æqui-curve Circles, or from the reciprocal proportion of the Radii of Curvature.

6. Therefore the Problem is reduced to this, that the Radius, or Center of Curvature may be found.

7. Imagine therefore that at three Points of the Curve δ , D, and d , Perpendiculars are drawn, of which those that are at D and δ meet in H, and those that are at D and d meet in b : And the Point D being in the middle, if there is a greater Curvity at the part D δ than at D d , then DH will be less than db . But by how much the Perpendiculars δ H and db are nearer the intermediate Perpendicular, so much the less will the distance be of the Points H and b : And at last when the Perpendiculars meet, those Points will coincide. Let them coincide in the Point C, then will C be the Center of Curvature, at the Point D of the Curve, on which the Perpendiculars stand; which is manifest of itself.



8. But there are several Symptoms or Properties of this Point C, which may be of use to its determination.

9. I. That it is the Concourse of Perpendiculars that are on each side at an infinitely little distance from DC.

10. II. That the Intersections of Perpendiculars, at any little finite distance on each side, are separated and divided by it; so that those which are on the more curved side D δ sooner meet at H, and those which are on the other less curved side D d meet more remotely at b .

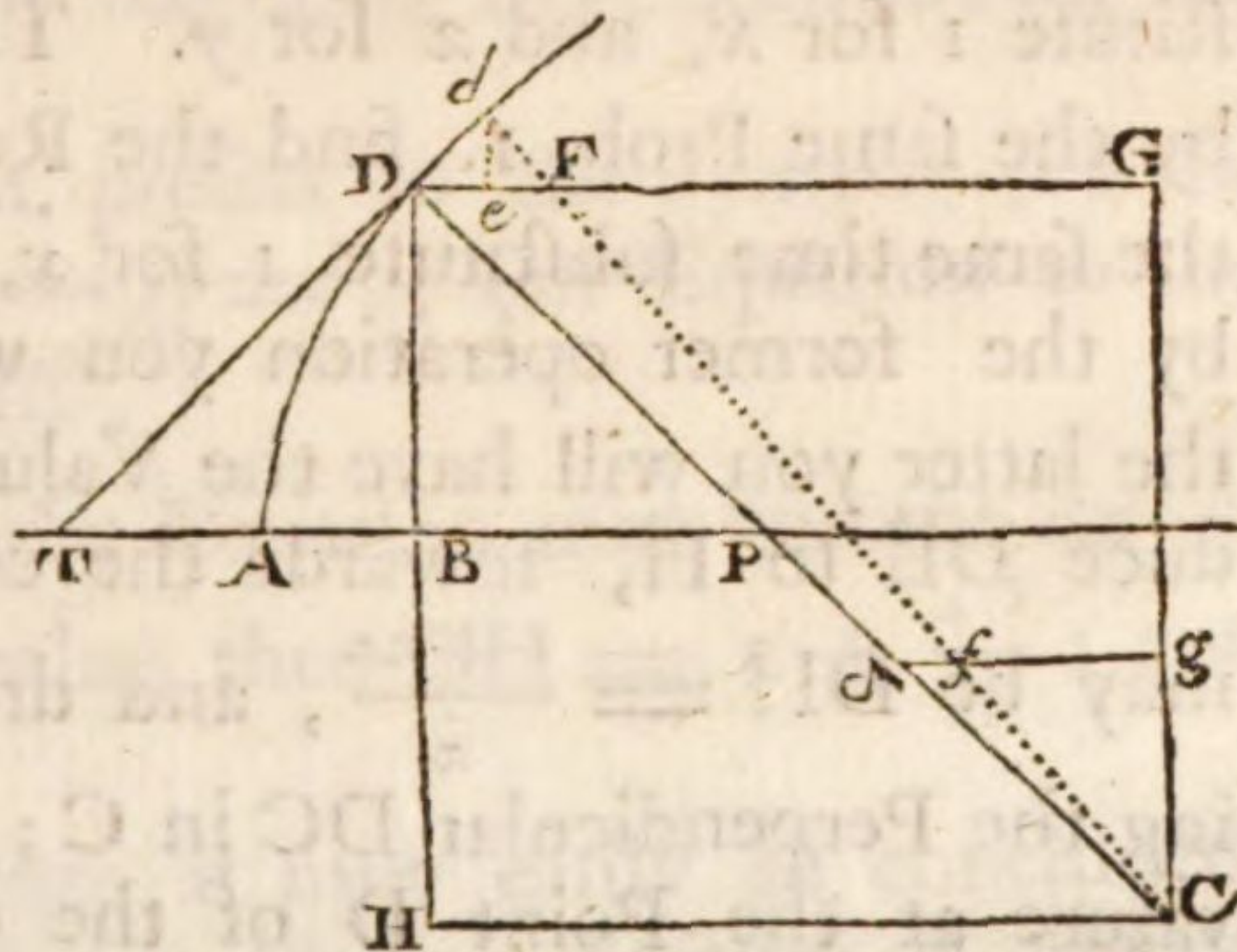
11. III. If DC be conceived to move, while it insists perpendicularly on the Curve, that point of it C, (if you except the motion of approaching to or receding from the Point of Infistence C,) will be least moved, but will be as it were the Center of Motion.

12. IV. If a Circle be described with the Center C, and the distance DC, no other Circle can be described, that can lie between at the Contact.

13. V. Lastly, if the Center H or *b* of any other touching Circle approaches by degrees to C the Center of this, till at last it coincides with it; then any of the points in which that Circle shall cut the Curve, will coincide with the point of Contact D.

14. And each of these Properties may supply the means of solving the Problem different ways: But we shall here make choice of the first, as being the most simple.

15. At any Point D of the Curve let DT be a Tangent, DC a Perpendicular, and C the Center of Curvature, as before. And let AB be the Absciss, to which let DB be apply'd at right Angles, and which DC meets in P. Draw



DG parallel to AB, and CG perpendicular to it, in which take Cg of any given Magnitude, and draw gδ perpendicular to it, which meets DC in δ. Then it will be $Cg : g\delta :: (TB : BD ::)$ the Fluxion of the Absciss, to the Fluxion of the Ordinate. Likewise imagine the Point D to move in the Curve an infinitely little distance D*d*, and

drawing *de* perpendicular to DG, and *Cd* perpendicular to the Curve, let *Cd* meet DG in F, and *δg* in *f*. Then will *De* be the Momentum of the Absciss, *de* the Momentum of the Ordinate, and *δf* the contemporaneous Momentum of the right Line *gδ*. Therefore $DF = De + \frac{de \times de}{De}$. Having therefore the Ratio's of these Moments, or, which is the same thing, of their generating Fluxions, you will have the Ratio of CG to the given Line Cg, (which is the same as that of DF to *δf*,) and thence the Point C will be determined.

16. Therefore let $AB = x$, $BD = y$, $Cg = 1$, and $g\delta = z$; then it will be $1 : z :: x : y$, or $z = \frac{y}{x}$. Now let the Mo-

mentum *δf* of *z* be $\dot{z} \times o$, (that is, the Product of the Velocity and of an infinitely small Quantity *o*,) and therefore the Moments

$De = \dot{x} \times o$, $de = \dot{y} \times o$, and thence $DF = \dot{x}o + \frac{\ddot{y}y o}{x}$. Therefore

'tis $Cg (1) : CG :: (\delta f : DF ::) \dot{z}o : \dot{x}o + \frac{\ddot{y}y o}{x}$. That is, $CG =$

$$\frac{\ddot{x}x + \ddot{y}y}{\dot{x}z}$$

17. And whereas we are at liberty to ascribe whatever Velocity we please to the Fluxion of the Absciss $\dot{x}$, (to which, as to an equable Fluxion, the rest may be referr'd;) make $\dot{x} = 1$, and then $\dot{y} = z$, and $CG = \frac{1+z\dot{z}}{z}$. And thence $DG = \frac{z+\dot{z}^2}{z}$, and $DC = \frac{1+z\dot{z}\sqrt{1+z\dot{z}}}{z}$.

18. Therefore any Equation being proposed, in which the Relation of BD to AB is express'd for defining the Curve; first find the Relation betwixt $\dot{x}$ and $\dot{y}$, by Prob. 1. and at the same time substitute 1 for $\dot{x}$, and z for $\dot{y}$. Then from the Equation that arises, by the same Prob. 1. find the Relation between $\dot{x}$, $\dot{y}$, and $\dot{z}$, and at the same time substitute 1 for $\dot{x}$, and z for $\dot{y}$, as before. And thus by the former operation you will obtain the Value of $\dot{z}$, and by the latter you will have the Value of z ; which being obtain'd, produce DB to H, towards the concave part of the Curve, that it may be $DH = \frac{1+z\dot{z}}{z}$, and draw HC parallel to AB, and meeting the Perpendicular DC in C; then will C be the Center of Curvature at the Point D of the Curve. Or since it is $1+z\dot{z} = \frac{PT}{BT}$, make $DH = \frac{PT}{z \times BT}$, or $DC = \frac{\overline{DP}^{\frac{1}{3}}}{z \times \overline{DB}^{\frac{1}{3}}}$.

19. Ex. 1. Thus the Equation $ax + bx^2 - y^2 = 0$ being proposed, (which is an Equation to the Hyperbola whose Latus rectum is a , and Transversum $\frac{a}{b}$;) there will arise (by Prob. 1.) $a + 2bx - 2zy = 0$, (writing 1 for $\dot{x}$, and z for $\dot{y}$ in the resulting Equation, which otherwise would have been $a\dot{x} + 2b\dot{x}x - 2\dot{y}y = 0$;) and hence again there arises $2b - 2z\dot{z} - 2zy = 0$, (1 and z being again wrote for $\dot{x}$ and $\dot{y}$.) By the first we have $z = \frac{a+2bx}{2y}$, and by the latter $\dot{z} = \frac{b-z\dot{z}}{y}$. Therefore any Point D of the Curve being given, and consequently x and y , from thence z and $\dot{z}$ will be given, which being known, make $\frac{1+z\dot{z}}{z} = GC$ or DH , and draw HC.

20. As if definitely you make $a = 3$, and $b = 1$, so that $3x + xx = yy$ may be the condition of the Hyperbola. And if you assume $x = 1$, then $y = 2$, $z = \frac{5}{4}$, $\dot{z} = -\frac{9}{8}$, and $DH = -9\frac{1}{8}$. H being found, raise the Perpendicular HC meeting the Perpendicular

cular DC before drawn ; or, which is the same thing, make HD : HC :: (1 : z ::) 1 : $\frac{z}{4}$. Then draw DC the Radius of Curvature.

21. When you think the Computation will not be too perplex, you may substitute the indefinite Values of $\dot{z}$ and z into $\frac{1+z\dot{z}}{\dot{z}}$, the Value of CG. Thus in the present Example, by a due Reduction you will have $DH = y + \frac{4\dot{y}^3 + 4by^3}{aa}$. Yet the Value of DH by Calculation comes out negative, as may be seen in the numeral Example. But this only shews, that DH must be taken towards B ; for if it had come out affirmative, it ought to have been drawn the contrary way.

22. COROL. Hence let the Sign prefixt to the Symbol $+b$ be changed, that it may be $ax - bxx - yy = 0$, (an Equation to the Ellipsis,) then $DH = y + \frac{4y^3 - 4by^3}{aa}$.

23. But supposing $b = 0$, that the Equation may become $ax - yy = 0$, (an Equation to the Parabola,) then $DH = y + \frac{4y^3}{aa}$; and thence $DG = \frac{1}{2}a + 2x$.

24. From these several Expressions it may easily be concluded, that the Radius of Curvature of any Conick Section is always $\frac{4DP^3}{aa}$.

25. Ex. 2. If $x^3 = ay^2 - xy^2$ be proposed, (which is the Equation to the Cissoïd of Diocles,) by Prob. 1. it will be first $3x^2 = 2axy - 2xzy - y^2$; and then $6x = 2axy + 2axz - 2zy - 2xzy - 2xzz - 2zy$: So that $z = \frac{3xx + yy}{2ay - 2xy}$, and $\dot{z} = \frac{3x - axz + 2zy + xzz}{ay - xy}$. Therefore any Point of the Cissoïd being given, and thence x and y , there will be given also z and $\dot{z}$; which being known, make $\frac{1+z\dot{z}}{\dot{z}} = CG$.

26. Ex. 3. If $b + y\sqrt{cc - yy} = xy$ were given, (which is the Equation to the Conchoid, in pag. 48 ;) make $\sqrt{cc - yy} = v$, and there will arise $bv + yv = xy$. Now the first of these, ($cc - yy = vv$,) will give (by Prob. 1.) $-2yz = 2vv$, (writing z for y ;) and the latter will give $bv + yv + zv = y + xz$. And from these Equations rightly disposed v and z will be determined. But that $\dot{z}$ may also be found ; out of the last Equation exterminate the Fluxion $\dot{v}$, by substituting $-\frac{yz}{v}$, and there will arise $-\frac{bvz}{v} - \frac{yz}{v} + zv = y$

$= y + xz$, an Equation that comprehends the flowing Quantities, without any of their Fluxions, as the Resolution of the first Problem requires. Hence therefore by Prob. 1. we shall have —

$$\frac{bx^2}{v} - \frac{byz}{v} + \frac{byzv}{vv} - \frac{zyxz}{v} - \frac{yyz}{v} + \frac{yyzv}{vv} + \dot{z}v + z\dot{v} = 2z + x\dot{z}.$$

This Equation being reduced, and disposed in order, will give $\dot{z}$. But when z and $\dot{z}$ are known, make $\frac{1+z\dot{z}}{\dot{z}} = CG$.

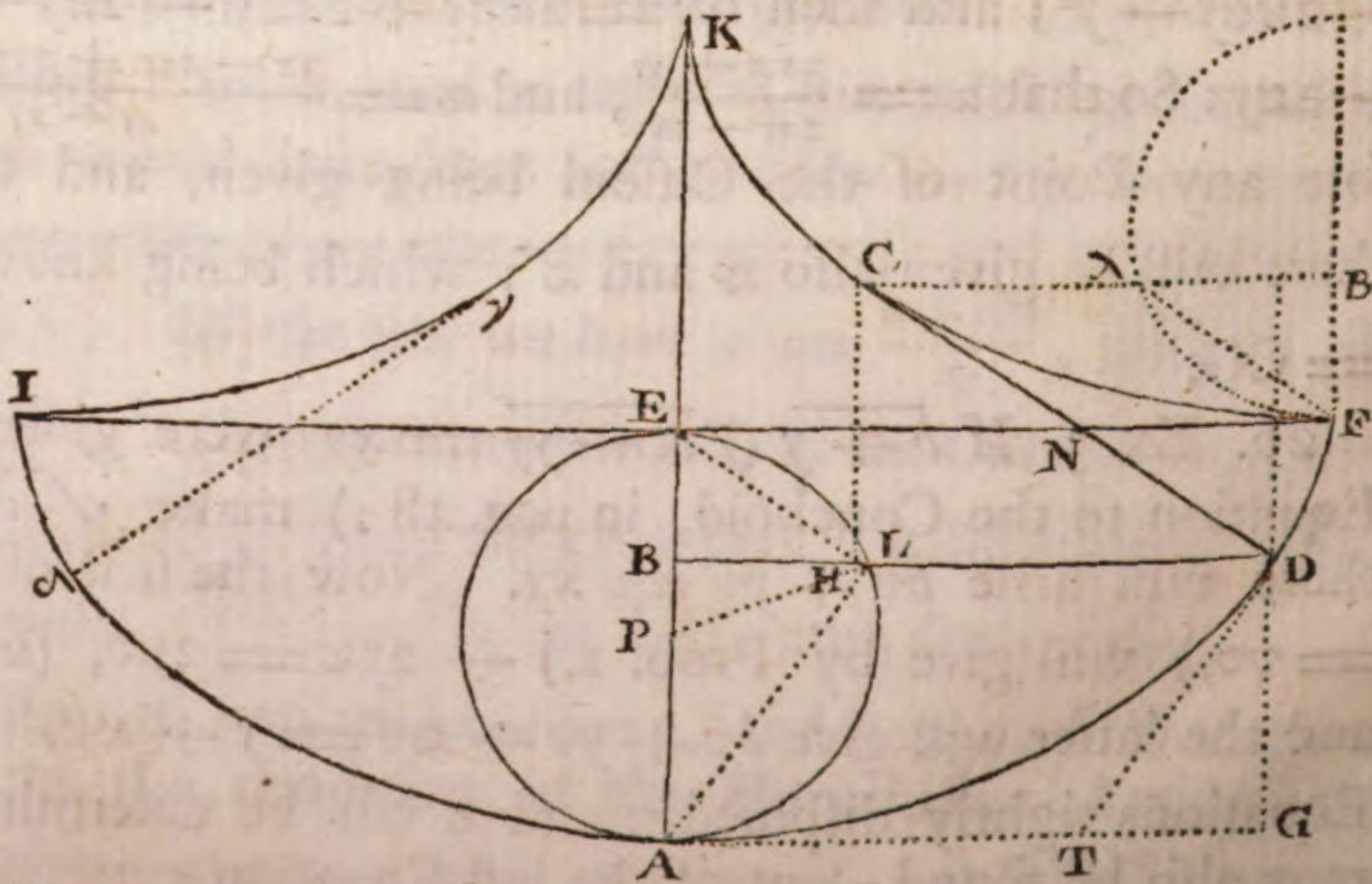
27. If we had divided the last Equation but one by z , then by Prob. 1. we should have had $-\frac{bz}{v} + \frac{byv}{vv} - \frac{zyz}{v} + \frac{yyv}{vv} + \dot{v} = 2 - \frac{yz}{xz}$; which would have been a more simple Equation than the former, for determining $\dot{z}$.

28. I have given this Example, that it may appear, how the operation is to be perform'd in surd Equations: But the Curvature of the Conchoid may be thus found a shorter way. The parts of the Equation $b + y\sqrt{cc - yy} = xy$ being squared, and divided by yy , there arises $\frac{b^2c^2}{y^2} + \frac{2bc^2}{y} - b^2 - 2by - y^2 = x^2$, and thence by Prob. 1.

$$-\frac{2b^2c^2z}{y^3} - \frac{2bc^2z}{y^2} - 2bz - 2yz = 2x, \text{ or } -\frac{b^2c^2}{y^3} - \frac{bc^2}{y^2} - b - y = \frac{x}{z}.$$

And hence again by Prob. 1. $\frac{3b^2c^2z}{y^4} + \frac{2bc^2z}{y^3} - z = \frac{1}{z} - \frac{x\dot{z}}{xz}$. By the first result z is determined, and $\dot{z}$ by the latter.

29. Ex. 4. Let ADF be a Trochoid [or Cycloid] belonging to the Circle ALE, whose Diameter is AE; and making the Ordinate BD to cut the Circle in L, call $AE = a$, $AB = x$, $BD = y$, $BL = v$, and the Arch $AL = t$, and the Fluxion of the same Arch $= \dot{t}$. And first (drawing the Semidiameter PL) the Fluxion of the



Base or Absciss AB will be to the Fluxion of the Arch AL, as BL to

to PL; that is, $\dot{x}$ or $1 : \dot{t} :: v : \frac{1}{2}a$. And therefore $\frac{a}{2v} = \dot{t}$. Then from the nature of the Circle $ax - xx = vv$, and therefore by Prob. 1. $a - 2x = 2\dot{v}$, or $\frac{a-2x}{2v} = \dot{v}$.

30. Moreover from the nature of the Trochoid, 'tis $LD = \text{Arch } AL$, and therefore $v + t = y$. And thence (by Prob. 1) $v + t = z$. Lastly, instead of the Fluxions $\dot{v}$ and $\dot{t}$ let their Values be substituted, and there will arise $\frac{a-x}{v} = z$. Whence (by Prob. 1.) is derived $-\frac{a\dot{v}}{vv} + \frac{x\dot{v}}{vv} - \frac{1}{v} = \dot{z}$. And these being found, make $\frac{1+z}{z} = -DH$, and raise the perpendicular HC.

31. COR. 1. Now it follows from hence, that $DH = 2BL$, and $CH = 2BE$, or that EF bisects the radius of Curvature CD in N. And this will appear by substituting the values of z and $\dot{z}$ now found, in the Equation $\frac{1+z}{z} = DH$, and by a proper reduction of the result.

32. COR. 2. Hence the Curve FCK, described indefinitely by the Center of Curvature of ADF, is another Trochoid equal to this, whose Vertices at I and F adjoin to the Cuspids of this. For let the Circle F λ , equal and alike posited to ALE, be described, and let C β be drawn parallel to EF, meeting the Circle in λ : Then will Arch F $\lambda = (\text{Arch } EL = NF =) C\lambda$.

33. COR. 3. The right Line CD, which is at right Angles to the Trochoid IAF, will touch the Trochoid IKF in the point C.

34. COR. 4. Hence (in the inverted Trochoids,) if at the Cuspid K of the upper Trochoid, a Weight be hung by a Thread at the distance KA or 2EA, and while the Weight vibrates, the Thread be suppos'd to apply itself to the parts of the Trochoid KF and KI, which resist it on each side, that it may not be extended into a right Line, but compel it (as it departs from the Perpendicular) to be by degrees inflected above, into the Figure of the Trochoid, while the lower part CD, from the lowest Point of Contact, still remains a right Line: The Weight will move in the Perimeter of the lower Trochoid, because the Thread CD will always be perpendicular to it.

35. COR. 5. Therefore the whole Length of the Thread KA is equal to the Perimeter of the Trochoid KCF, and its part CD is equal to the part of the Perimeter CF.

36. COR. 6. Since the Thread by its oscillating Motion revolves about the moveable Point C, as a Center; the Superficies through which the whole Line CD continually passes, will be to the Superficies through which the part CN above the right Line IF passes at the same time, as $\overline{CD}^2$ to $\overline{CN}^2$, that is, as 4 to 1. Therefore the Area CFN is a fourth part of the Area CFD; and the Area KCNE is a fourth part of the Area AKCD.

37. COR. 7. Also since the subtense EL is equal and parallel to CN, and is converted about the immoveable Center E, just as CN moves about the moveable Center C; the Superficies will be equal through which they pass in the same time, that is, the Area CFN, and the Segment of the Circle EL. And thence the Area NFD will be the triple of that Segment, and the whole area EADF will be the triple of the Semicircle.

38. COR. 8. When the Weight D arrives at the point F, the whole Thread will be wound about the Perimeter of the Trochoid KCF, and the Radius of Curvature will there be nothing. Wherefore the Trochoid IAF is more curved, at its Cuspid F, than any Circle; and makes an Angle of Contact, with the Tangent βF produced, infinitely greater than a Circle can make with a right Line.

39. But there are Angles of Contact that are infinitely greater than Trochoidal ones, and others infinitely greater than these, and so on *in infinitum*; and yet the greatest of them all are infinitely less than right-lined Angles. Thus $xx = ay$, $x^3 = by^2$, $x^4 = cy^3$, $x^5 = dy^4$, &c. denote a Series of Curves, of which every succeeding one makes an Angle of Contact with its Absciss, which is infinitely greater than the preceding can make with the same Absciss. And the Angle of Contact which the first $xx = ay$ makes, is of the same kind with Circular ones; and that which the second $x^3 = by^2$ makes, is of the same kind with Trochoidals. And tho' the Angles of the succeeding Curves do always infinitely exceed the Angles of the preceding, yet they can never arrive at the magnitude of a right-lined Angle.

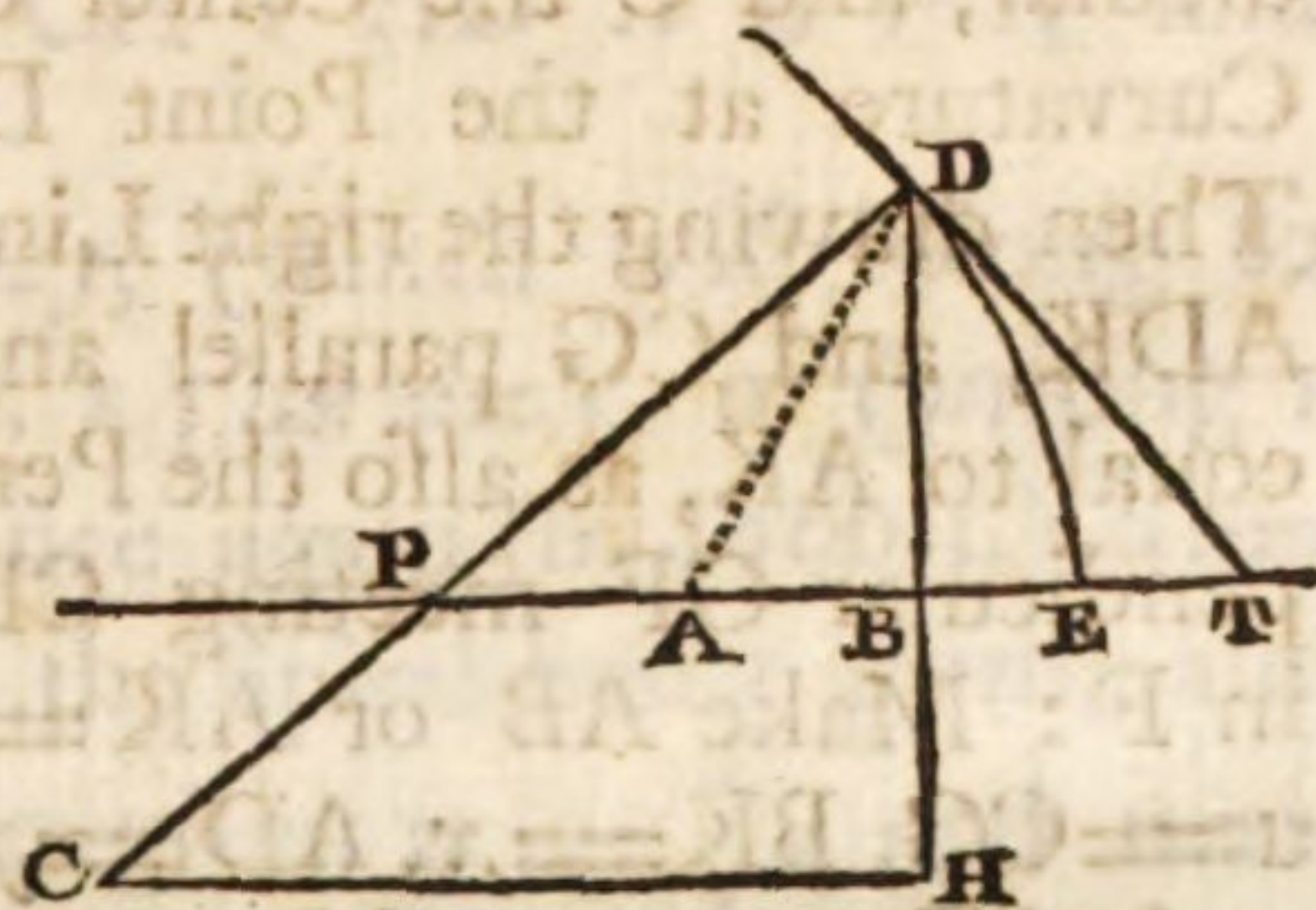
40. After the same manner $x = y$, $xx = ay$, $x^3 = b^2y$, $x^4 = c^3y$, &c. denote a Series of Lines, of which the Angles of the subsequents, made with their Absciss's at the Vertices, are always infinitely less than the Angles of the preceding. Moreover, between the Angles of Contact of any two of these kinds, other Angles of Contact may be found *ad infinitum*, that shall infinitely exceed each other.

41. Now it appears, that Angles of Contact of one kind are infinitely greater than those of another kind; since a Curve of one kind, however great it may be, cannot, at the Point of Contact,

lie between the Tangent and a Curve of another kind, however small that Curve may be. Or an Angle of Contact of one kind cannot necessarily contain an Angle of Contact of another kind, as the whole contains a part. Thus the Angle of Contact of the Curve $x^4 = cy^3$, or the Angle which it makes with its Absciss, necessarily includes the Angle of Contact of the Curve $x^3 = by^2$, and can never be contain'd by it. For Angles that can mutually exceed each other are of the same kind, as it happens with the aforesaid Angles of the Trochoid, and of this Curve $x^3 = by^2$.

42. And hence it appears, that Curves, in some Points, may be infinitely more straight, or infinitely more curved, than any Circle, and yet not, on that account, lose the form of Curve-lines. But all this by the way only.

43. Ex. 5. Let ED be the Quadratrix to the Circle, described from Center A; and letting fall DB perpendicular to AE, make $AB = x$, $BD = y$, and $AE = 1$. Then 'twill be $yx - yy^2 - yx^2 = xy$, as before. Then writing 1 for x , and z for y , the Equation becomes $zx - zy^2 - zx^2 = y$; and thence, by Prob. 1. $zx - zy^2 - zx^2 + zx - 2zx^2 - 2zyy = y$. Then reducing, and again writing 1 for x , and z for y , there arises $z = \frac{2z^2y + 2zx}{x - xx - yy}$. But z and z being found, make $\frac{1 + zz}{z} = DH$, and draw HC as above.



44. If you desire a Construction of the Problem, you will find it very short. Thus draw DP perpendicular to DT, meeting AT in P, and make $2AP : AE :: PT : CH$. For $z = \left(\frac{y}{x - xx - yy} \right) \frac{BD}{BT}$, and $zy = \frac{BDq}{BT} = -BP$; and $zy + x = -AP$, and $\frac{2z}{x - xx - yy}$ into $zy + x = \frac{2BD}{AE \times BTq}$ into $-AP = z$. Moreover it is $1 + zz = \frac{PT}{BT}$, (because $1 + \frac{BDq}{BTq} = \frac{DTq}{BTq}$), and therefore $\frac{1 + zz}{z} = \frac{PT \times AE \times BT}{-2BD \times AP} = DH$. Lastly, it is $BT : BD :: DH : CH = \frac{PT \times AE}{-2AP}$. Here the negative Value only shews, that CH must be taken the same way as AB from DH.

45. In the same manner the Curvature of Spirals, or of any other Curves whatever, may be determined by a very short Calculation.

that in this kind of Computations, I take those Quantities (AD and Ae) for equal, the Ratio of which differs but infinitely little from the Ratio of Equality.

49. Now from hence arises the following Rule. The Relation of x and y being exhibited by any Equation, find the Relation of the Fluxions $\dot{x}$ and $\dot{y}$, (by Prob. 1.) and substitute 1 for $\dot{x}$, and $y\dot{z}$ for $\dot{y}$. Then from the resulting Equation find again, (by Prob. 1.) the Relation between $\dot{x}$, $\dot{y}$, and $\dot{z}$, and again substitute 1 for $\dot{x}$. The first result by due reduction will give $\dot{y}$ and $\dot{z}$, and the latter will give $\dot{x}$; which being known, make $\frac{y + y\dot{z}\dot{z}}{1 + \dot{z}\dot{z} - \dot{z}} = DH$, and raise the Perpendicular HC, meeting the Perpendicular to the Spiral DC before drawn in C, and C will be the Center of Curvature. Or which comes to the same thing, take $CH : HD :: \dot{z} : -1$, and draw CD.

50. Ex. 1. If the Equation be $ax = y$, (which will belong to the Spiral of *Archimedes*,) then (by Prob. 1.) $a\dot{x} = \dot{y}$, or (writing 1 for $\dot{x}$, and $y\dot{z}$ for $\dot{y}$,) $a = y\dot{z}$. And hence again (by Prob. 1.) $0 = y\dot{z} + y\dot{z}$. Wherefore any Point D of the Spiral being given, and thence the length AD or y , there will be given $\dot{z} = \frac{a}{y}$, and $\dot{x} = (-\frac{y\dot{z}}{y} \text{ or } -\frac{a\dot{z}}{y})$. Which being known, make $1 + \dot{z}\dot{z} - \dot{z} : 1 + \dot{z}\dot{z} :: DA (y) : DH$. And $1 : \dot{z} :: DH : CH$.

And hence you will easily deduce the following Construction. Produce AB to Q, so that $AB : \text{Arch BK} :: \text{Arch BK} : BQ$, and make $AB + AQ : AQ :: DA : DH :: a : HC$.

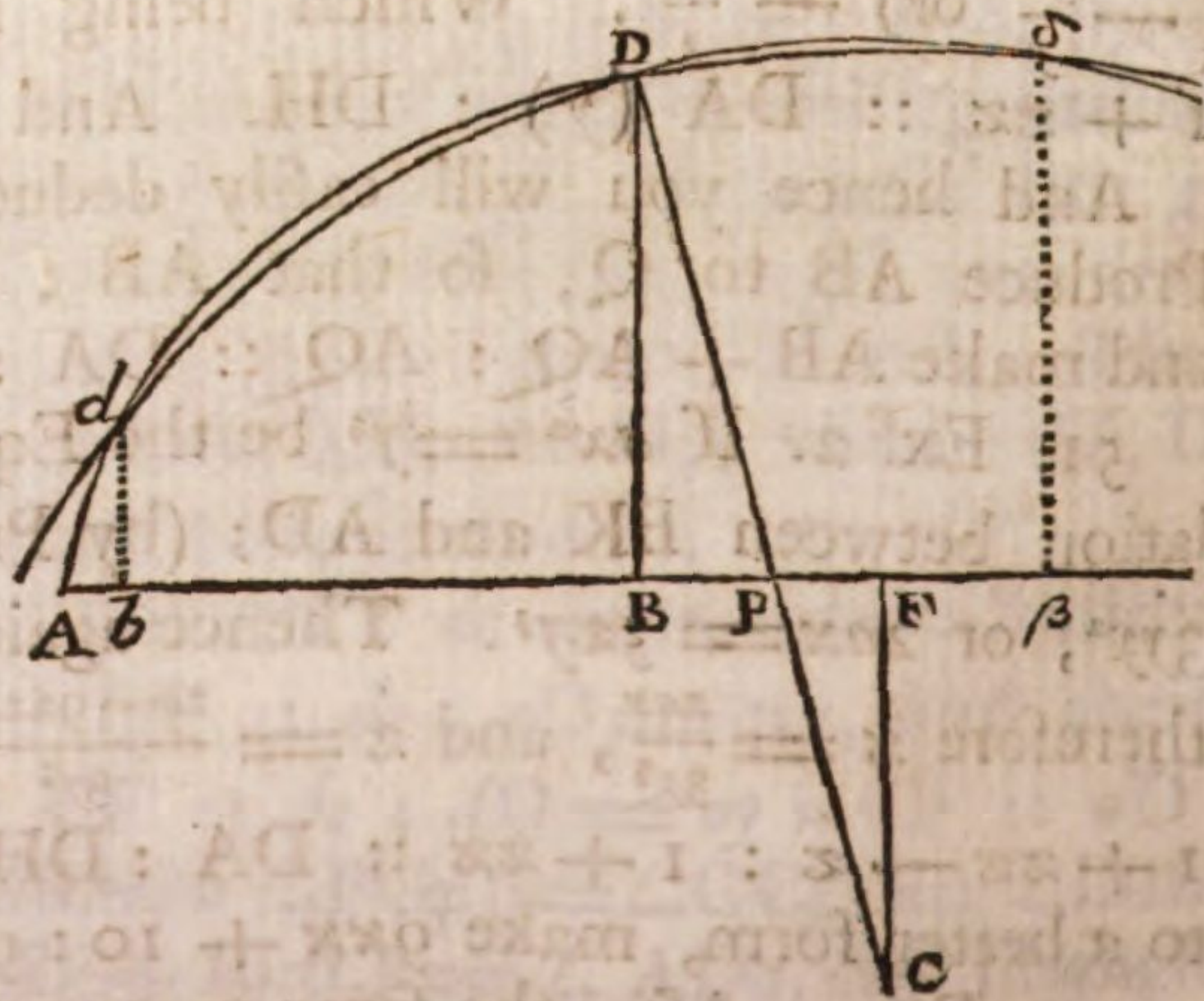
51. Ex. 2. If $ax^2 = y^3$ be the Equation that determines the Relation between BK and AD; (by Prob. 1.) you will have $2a\dot{x}\dot{x} = 3y\dot{y}^2$, or $2a\dot{x} = 3\dot{z}y^2$. Thence again $2a\dot{x} = 3\dot{z}y^2 + 9\dot{z}y\dot{y}$. 'Tis therefore $\dot{z} = \frac{2a\dot{x}}{3y^2}$, and $\dot{x} = \frac{2a - 9\dot{z}y^2}{3y^2}$. These being known, make $1 + \dot{z}\dot{z} - \dot{z} : 1 + \dot{z}\dot{z} :: DA : DH$. Or, the work being reduced to a better form, make $9\dot{x}\dot{x} + 10 : 9\dot{x}\dot{x} + 4 :: DA : DH$.

52. Ex. 3. After the same manner, if $ax^2 - bxy = y^3$ determines the Relation of BK to AD; there will arise $\frac{2ax - by}{bxy + 3y^2} = \dot{z}$, and $\frac{2a - 2b\dot{z}y - b\dot{z}^2xy - 9\dot{z}^2y^2}{bxy + 3y^2} = \dot{x}$. From which DH, and thence the Point C, is determined as before.

53. And thus you will easily determine the Curvature of any other Spirals ; or invent Rules for any other kinds of Curves, in imitation of these already given.

54. And now I have finish'd the Problem ; but having made use of a Method which is pretty different from the common ways of operation, and as the Problem itself is of the number of those which are not very frequent among Geometricians : For the illustration and confirmation of the Solution here given, I shall not think much to give a hint of another, which is more obvious, and has a nearer relation to the usual Methods of drawing Tangents. Thus if from any Center, and with any Radius, a Circle be conceived to be described, which may cut any Curve in several Points ; if that Circle be suppos'd to be contracted, or enlarged, till two of the Points of intersection coincide, it will there touch the Curve. And besides, if its Center be suppos'd to approach towards, or recede from, the Point of Contact, till the third Point of intersection shall meet with the former in the Point of Contact ; then will that Circle be æquicurved with the Curve in that Point of Contact : In like manner as I insinuated before, in the last of the five Properties of the Center of Curvature, by the help of each of which I affirm'd the Problem might be solved in a different manner.

55. Therefore with Center C, and Radius CD, let a Circle be described, that cuts the Curve in the Points d , D, and δ ; and letting fall the Perpendiculars DB, db , $\delta\beta$, and CF, to the Absciss AB ; call AB $=x$, BD $=y$, AF $=v$, FC $=t$, and DC $=s$. Then BF $=v-x$, and DB+FC $=y+t$. The sum of the Squares of these is equal to the Square of DC ; that is, $v^2 - 2vx + x^2 + y^2 + 2yt + t^2 = s^2$. If you would abbreviate this, make $v^2 + t^2 - s^2 = q^2$, (any Symbol at pleasure,) and it becomes $x^2 - 2vx + y^2 + 2ty + q^2 = 0$. After you have found t , v , and q^2 , you will have $s = \sqrt{v^2 + t^2 - q^2}$.



56. Now let any Equation be proposed for defining the Curve, the quantity of whose Curvature is to be found. By the help of this Equation you may exterminate either of the Quantities x or y ,
and

and there will arise an Equation, the Roots of which, (db , DB , $\delta\beta$, &c. if you exterminate x ; or Ab , AB , $A\beta$, &c. if you exterminate y ;) are at the Points of intersection d , D , δ , &c. Wherefore since three of them become equal, the Circle both touches the Curve, and will also be of the same degree of Curvature as the Curve, in the point of Contact. But they will become equal by comparing the Equation with another fictitious Equation of the same number of Dimensions, which has three equal Roots; as *Des Cartes* has shew'd. Or more expeditiously by multiplying its Terms twice by an Arithmetical Progression.

57. EXAMPLE. Let the Equation be $ax = yy$, (which is an Equation to the Parabola,) and exterminating x , (that is, substituting its Value $\frac{y}{a}$ in the foregoing Equation,) there will arise

Three of whose Roots y are to be made equal. And for this purpose I multiply the Terms twice by an Arithmetical Progression, as you see done here; and there arises

$$\begin{array}{r} \frac{y^4}{aa} * - \frac{2y}{a} y^2 + 2ty + q^2 = 0. \\ + y^2 \\ 4 * 2 \quad 1 \quad 0 \\ 3 * 1 \quad 0 \quad -1 \\ \hline \frac{12y^4}{aa} - \frac{4y}{a} y^2 + 2y^2 = 0. \end{array}$$

Or $v = \frac{3y^2}{a} + \frac{1}{2}a$. Whence it is easily infer'd, that $BF = 2x + \frac{1}{2}a$, as before.

58. Wherefore any Point D of the Parabola being given, draw the Perpendicular DP to the Curve, and in the Axis take $PF = 2AB$, and erect FC Perpendicular to FA , meeting DP in C ; then will C be the Center of Curvity desired.

59. The same may be perform'd in the Ellipsis and Hyperbola, but the Calculation will be troublesome enough, and in other Curves generally very tedious.

Of Questions that have some Affinity to the preceding Problem.

60. From the Resolution of the preceding Problem some others may be perform'd; such are,

I. To find the Point where the Curve has a given degree of Curvature.

61. Thus in the Parabola, $ax = yy$, if the Point be required whose Radius of Curvature is of a given length f : From the Center of Curvature, found as before, you will determine the Radius

to

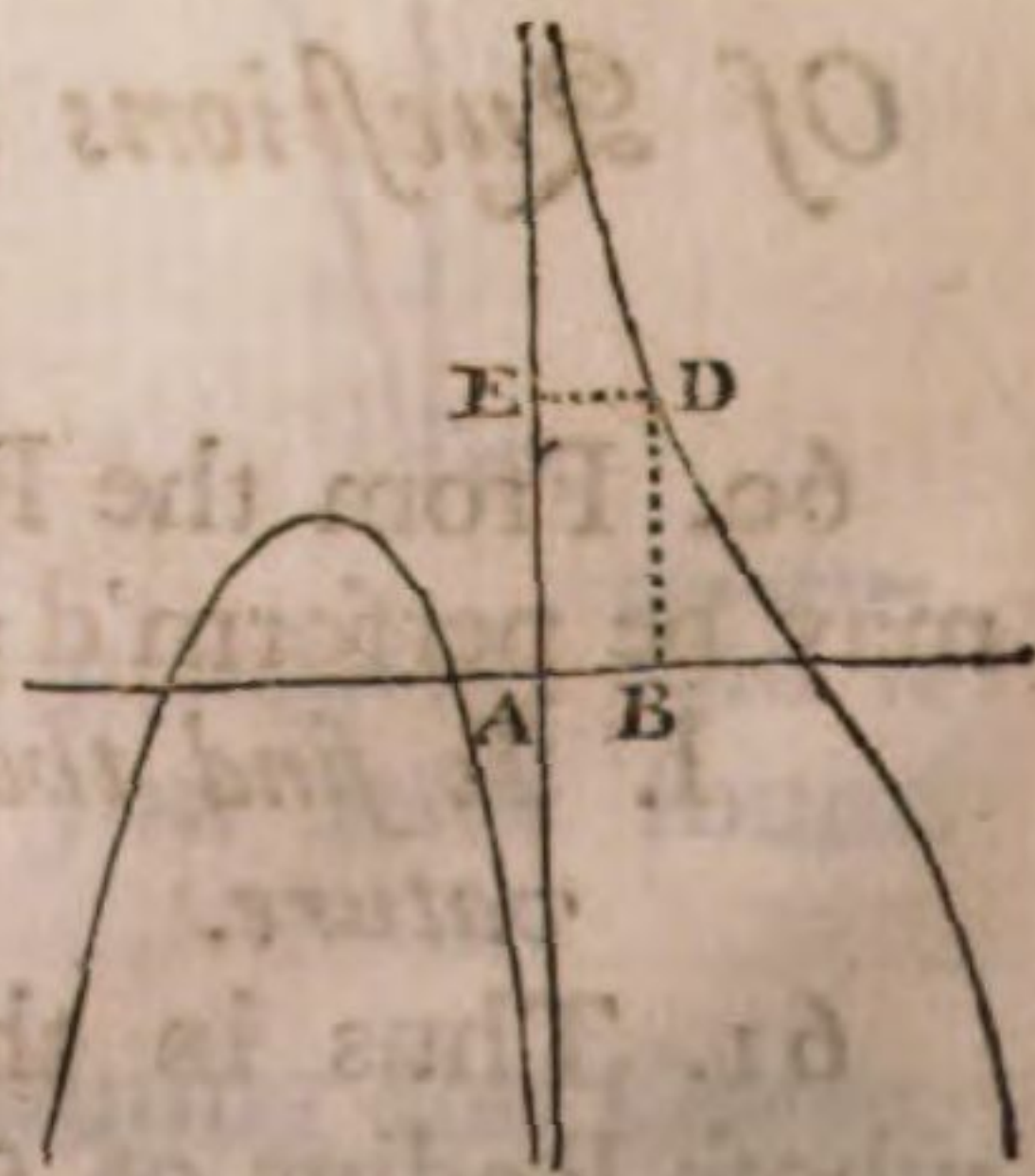
to be $\frac{a+4x}{2a} \sqrt{aa+4ax}$, which must be made equal to f . Then by reduction there arises $x = -\frac{1}{4}a + \sqrt[3]{\frac{1}{8}aff}$.

II. To find the Point of Rectitude.

62. I call that the *Point of Rectitude*, in which the Radius of Flexure becomes infinite, or its Center at an infinite distance: Such it is at the Vertex of the Parabola $a^3x = y^4$. And this same Point is commonly the Limit of contrary Flexure, whose Determination I have exhibited before. But another Determination, and that not inelegant, may be derived from this Problem. Which is, the longer the Radius of Flexure is, so much the less the Angle DCd (*Fig. pag. 61.*) becomes, and also the Moment δf ; so that the Fluxion of the Quantity z is diminish'd along with it, and by the Infinitude of that Radius, altogether vanishes. Therefore find the Fluxion z , and suppose it to become nothing.

63. As if we would determine the Limit of contrary Flexure in the Parabola of the second kind, by the help of which *Cartesius* constructed Equations of six Dimensions; the Equation to that Curve is $x^3 - bx^2 - cd x + bcd + dxy = 0$. And hence (by Prob. 1.) arises $3xx^2 - 2bxx - cd\dot{x} + d\dot{x}y + dx\dot{y} = 0$. Now writing 1 for $\dot{x}$, and z for $\dot{y}$, it becomes $3x^2 - 2bx - cd + dy + dxz = 0$; whence again (by Prob. 1.) $6xx - 2b\dot{x} + \dot{d}y + dx\dot{z} + dxz = 0$. Here again writing 1 for $\dot{x}$, z for $\dot{y}$, and 0 for $\dot{z}$, it becomes $6x - 2b + 2dz = 0$. And exterminating z , by putting $b - 3x$ for dz in the Equation $3xx - 2bx - cd + dy + dxz = 0$, there will arise $-bx - cd + dy = 0$, or $y = c + \frac{bx}{d}$; this being substituted in the room of y in the Equation of the Curve, we shall have $x^3 + bcd = 0$; which will determine the Confine of contrary Flexure.

64. By a like Method you may determine the Points of Rectitude, which do not come between parts of contrary Flexure. As if the Equation $x^4 - 4ax^3 + 6a^2x^2 - b^3y = 0$ express'd the nature of a Curve; you have first, (by Prob. 1.) $4x^3 - 12ax^2 + 12a^2x - b^3z = 0$, and hence again $12x^2 - 24ax + 12a^2 - b^3\dot{z} = 0$. Here suppose $\dot{z} = 0$, and by Reduction there will arise $x = a$. Wherefore take $AB = a$, and erect the perpendicular BD ; this will meet the Curve in the Point of Rectitude D , as was required.



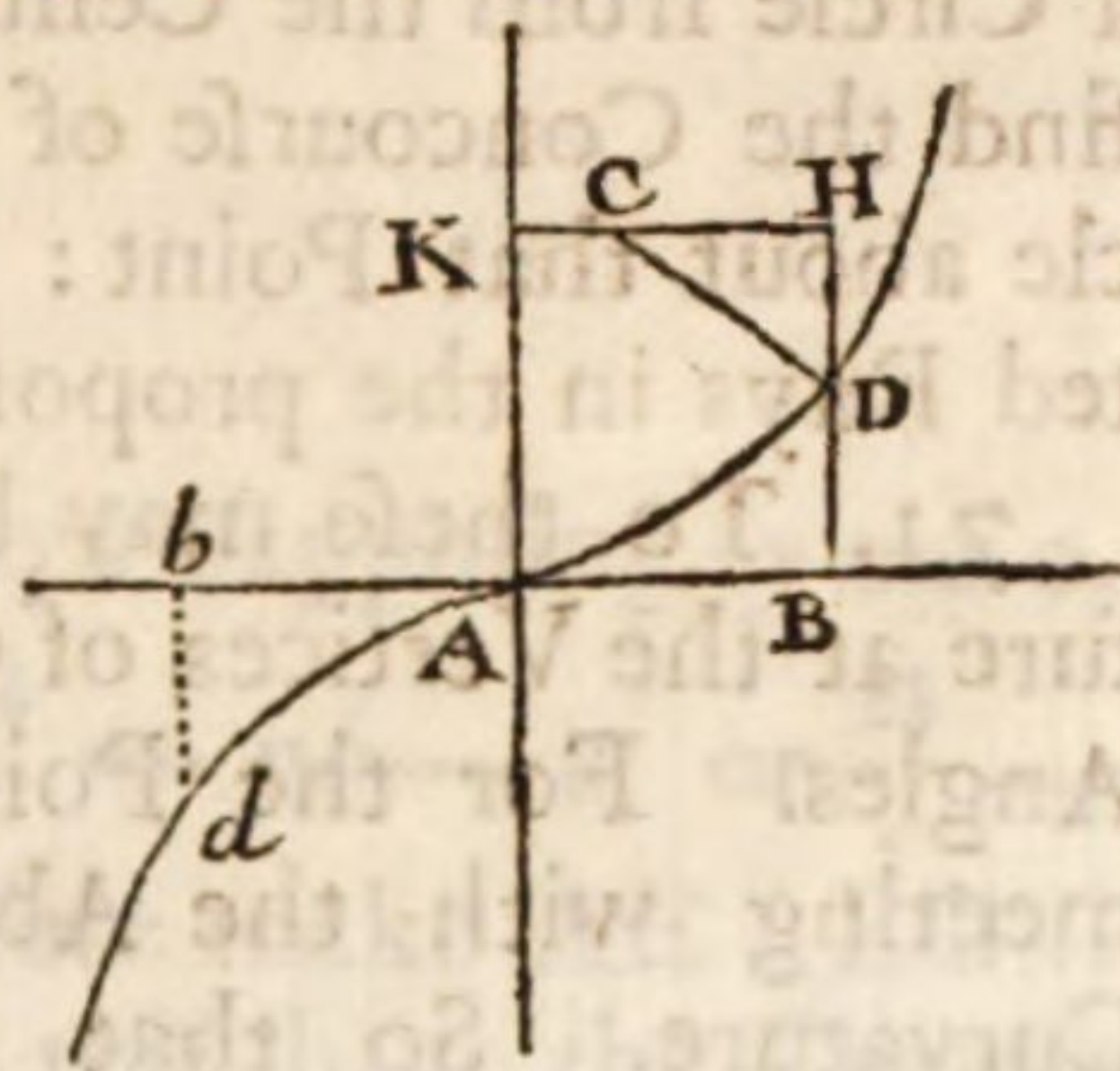
III. To find the Point of infinite Flexure.

65. Find the Radius of Curvature, and suppose it to be nothing. Thus to the Parabola of the second kind, whose Equation is $x^3 = ay^2$, that Radius will be $CD = \frac{4a+9x}{6a} \sqrt{4ax+9xx}$; which becomes nothing when $x = 0$.

IV. To determine the Point of the greatest or least Flexure.

66. At these Points the Radius of Curvature becomes either the greatest or least. Wherefore the Center of Curvature, at that moment of Time, neither moves towards the point of Contact, nor the contrary way, but is intirely at rest. Therefore let the Fluxion of the Radius CD be found; or more expeditiously, let the Fluxion of either of the Lines BH or AK be found, and let it be made equal to nothing.

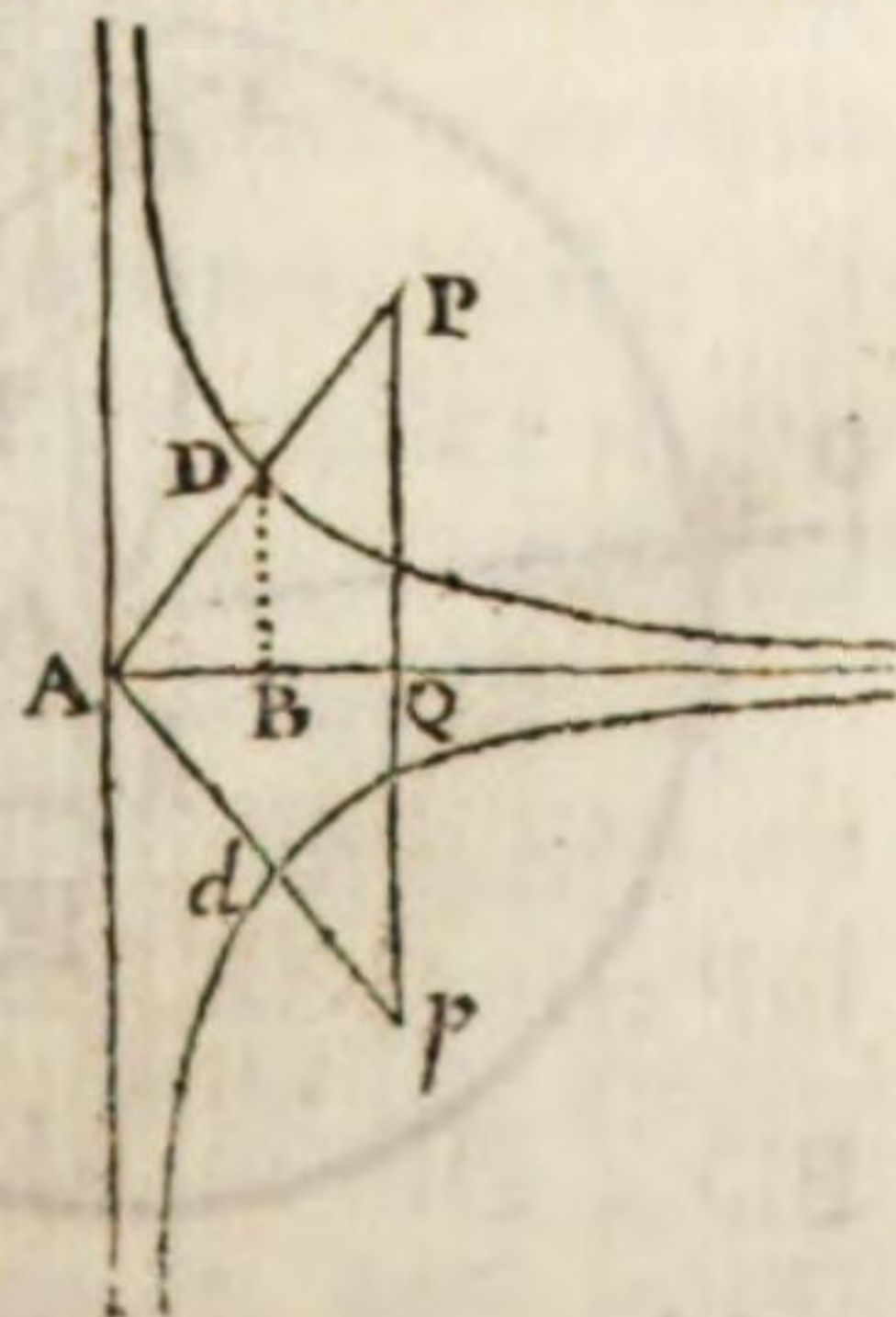
67. As if the Question were proposed concerning the Parabola of the second kind $x^3 = a^2y$; first to determine the Center of Curvature you will find $DH = \frac{aa+9xy}{6x}$,



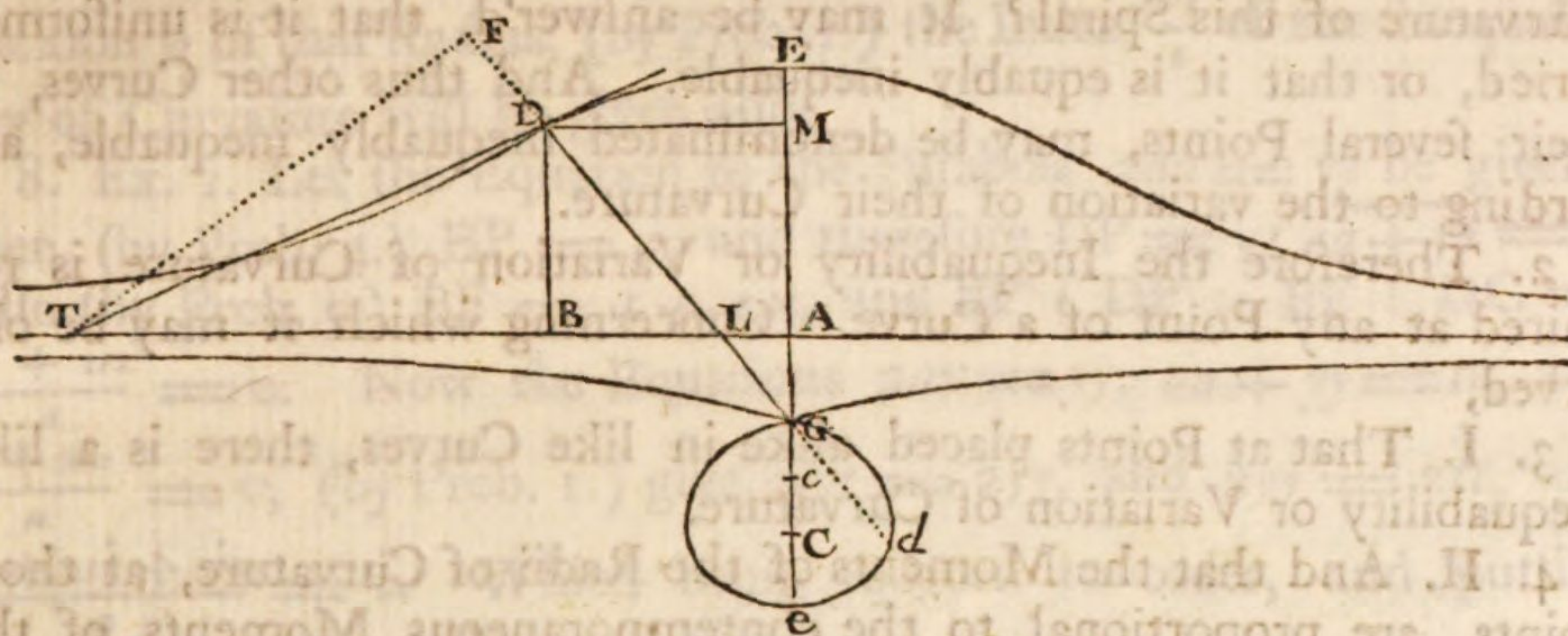
and therefore $BH = \frac{aa+15xy}{6x}$; make $BH = v$, then $\frac{aa}{6x} + \frac{5}{2}y = v$. Hence (by Prob. 1.) $-\frac{a^2x}{6xx} + \frac{5}{2}y = \dot{v}$. But now suppose $\dot{v}$, or the Fluxion of BH, to be nothing; and besides, since by Hypothesis $x^3 = a^2y$, and thence (by Prob. 1.) $3xx^2 = a^2\dot{y}$, putting $x = 1$, substitute $\frac{3xx}{aa}$ for $\dot{y}$, and there will arise $45x^4 = a^4$. Take therefore

$AB = a \sqrt[4]{\frac{45}{a^4}} = a \times \sqrt[4]{45}$, and raising the perpendicular BD, it will meet the Curve in the Point of the greatest Curvature. Or, which is the same thing, make $AB : BD :: 3\sqrt{5} : 1$.

68. After the same manner the Hyperbola of the second kind represented by the Equation $xy^2 = a^3$, will be most inflected in the points D and d, which you may determine by taking in the Absciss $AQ = 1$, and erecting the Perpendicular $QP = \sqrt{5}$, and Qp equal to it on the other side. Then drawing AP and Ap, they will meet the Curve in the points D and d required.



73. And in like manner for the Conchoid, defined by the Equation $\frac{b^2c^2}{xx} + \frac{2bcc}{x} - \frac{cc}{bb} - 2bx - xx = yy$, the Value of yy , (found by

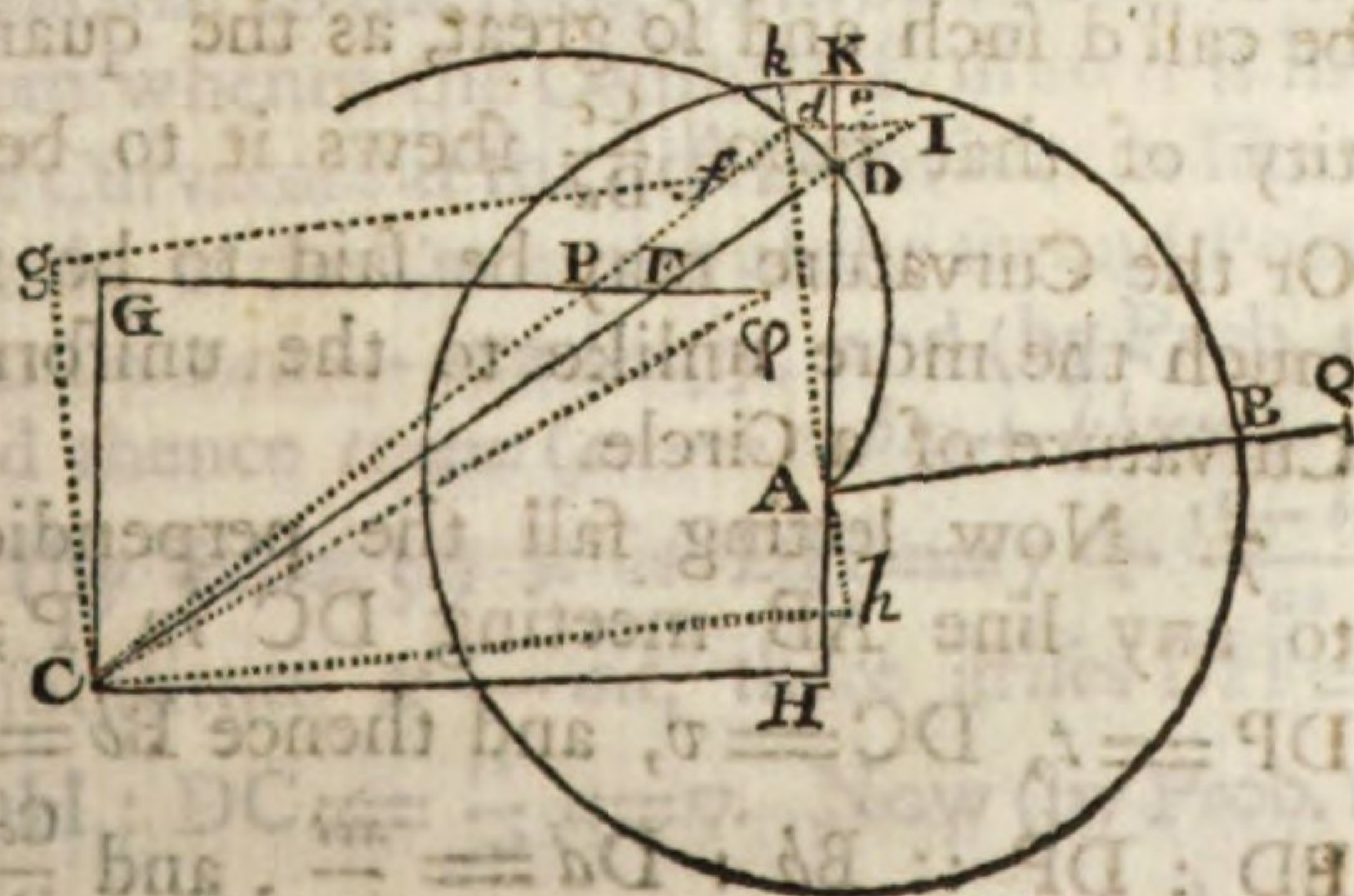


Prob. 1.) will be $-\frac{b^2c^2}{x^3} - \frac{bc^2}{x^2} - b - x$. Now supposing $y = 0$, and thence $x = c$ or $-c$, we shall have $-\frac{bb}{c} - 2b - c$, or $\frac{bb}{c} - 2b + c$, for the Radius of Curvature. Therefore make $AE : EG :: EG : EC$, and $Ae : eG :: eG : ec$, and you will have the Centers of Curvature C and c , at the Vertices E and e of the Conjugate Conchoids.

PROB. VI.

To determine the Quality of the Curvature, at a given Point of any Curve.

1. By the Quality of Curvature I mean its Form, as it is more or less inequable, or as it is varied more or less, in its progress thro' different parts of the Curve. So if it were demanded, what is the Quality of the Curvature of the Circle? it might be answer'd, that it is uniform, or invariable. And thus if it were demanded, what is the Quality of the Curvature of the Spiral, which is described by the motion of the point D , proceeding from A in AD with an accelerated velocity, while the right Line AK moves with an uniform rotation about the Center A ; the acceleration of



which Velocity is such, that the right Line AD has the same ratio to the Arch BK, described from a given point B, as a Number has to its Logarithm: I say, if it be ask'd, What is the Quality of the Curvature of this Spiral? It may be answer'd, that it is uniformly varied, or that it is equably inequable. And thus other Curves, in their several Points, may be denominated inequably inequable, according to the variation of their Curvature.

2. Therefore the Inequability or Variation of Curvature is required at any Point of a Curve. Concerning which it may be observed,

3. I. That at Points placed alike in like Curves, there is a like Inequability or Variation of Curvature.

4. II. And that the Moments of the Radii of Curvature, at those Points, are proportional to the contemporaneous Moments of the Curves, and the Fluxions to the Fluxions.

5. III. And therefore, that where those Fluxions are not proportional, the Inequability of the Curvature will be unlike. For there will be a greater Inequability, where the Ratio of the Fluxion of the Radius of Curvature to the Fluxion of the Curve is greater. And therefore that ratio of the Fluxions may not improperly be call'd the Index of the Inequability or of the Variation of Curvature.

6. At the points D and d , infinitely near to each other, in the Curve AD d , let there be drawn the Radii of Curvature DC and dc ; and D d being the Moment of the Curve, C c will be the contemporaneous Moment of the Radius of Curvature, and $\frac{C_c}{D_d}$ will be the Index of the Inequability of Curvature. For the Inequability may be call'd such and so great, as the quantity of that ratio $\frac{C_c}{D_d}$ shews it to be: Or the Curvature may be said to be so much the more unlike to the uniform Curvature of a Circle.

7. Now letting fall the perpendicular Ordinates DB and db , to any line AB meeting DC in P; make AB = x , BD = y , DP = t , DC = v , and thence B b = $x\dot{o}$, it will be C c = $v\dot{o}$; and BD : DP :: B b : D d = $\frac{x\dot{o}t}{y}$, and $\frac{C_c}{D_d} = \frac{v\dot{o}}{\frac{x\dot{o}t}{y}} = \frac{vy}{xt}$, making $x = 1$.

Wherefore



Wherefore the relation between x and y being exhibited by any Equation, and thence, (according to Prob. 4. and 5.) the Perpendicular DP or t , being found, and the Radius of Curvature v , and the Fluxion $\dot{v}$ of that Radius, (by Prob. 1.) the Index $\frac{\dot{v}y}{t}$ of the Inequability of Curvature will be given also.

8. Ex. 1. Let the Equation to the Parabola $2ax = yy$ be given; then (by Prob. 4.) $BP = a$, and therefore $DP = \sqrt{aa + yy} = t$. Also (by Prob. 5.) $BF = a + 2x$, and $BP : DP :: BF : DC = \frac{at + 2tx}{a} = v$. Now the Equations $2ax = yy$, $aa + yy = tt$, and $\frac{at + 2tx}{a} = v$, (by Prob. 1.) give $2\dot{a}x = 2\dot{y}y$, and $2\dot{y}y = 2\dot{t}t$, and $\frac{\dot{a}t + 2t\dot{x} + 2tx}{a} = \dot{v}$. Which being reduced to order, and putting $\dot{x} = 1$, there will arise $\dot{y} = \frac{a}{y}$, $\dot{t} = \left(\frac{\dot{y}}{t} = \right) \frac{a}{t}$, and $\dot{v} = \frac{\dot{a}t + 2t\dot{x} + 2tx}{a}$.

And thus $\dot{y}$, $\dot{t}$, and $\dot{v}$ being found, there will be had $\frac{\dot{v}y}{t}$ the Index of the Inequability of Curvature.

9. As if in Numbers it were determin'd, that $a = 1$, or $2x = yy$, and $x = \frac{1}{2}$; then $y (= \sqrt{2x}) = 1$, $\dot{y} (= \frac{a}{y}) = 1$, $t (= \sqrt{aa + yy}) = \sqrt{2}$, $\dot{t} (= \frac{a}{t}) = \frac{1}{\sqrt{2}}$, and $\dot{v} = \left(\frac{\dot{a}t + 2t\dot{x} + 2tx}{a} = \right) 3\sqrt{2}$. So that $\frac{\dot{v}y}{t} = 3$, which therefore is the Index of Inequability.

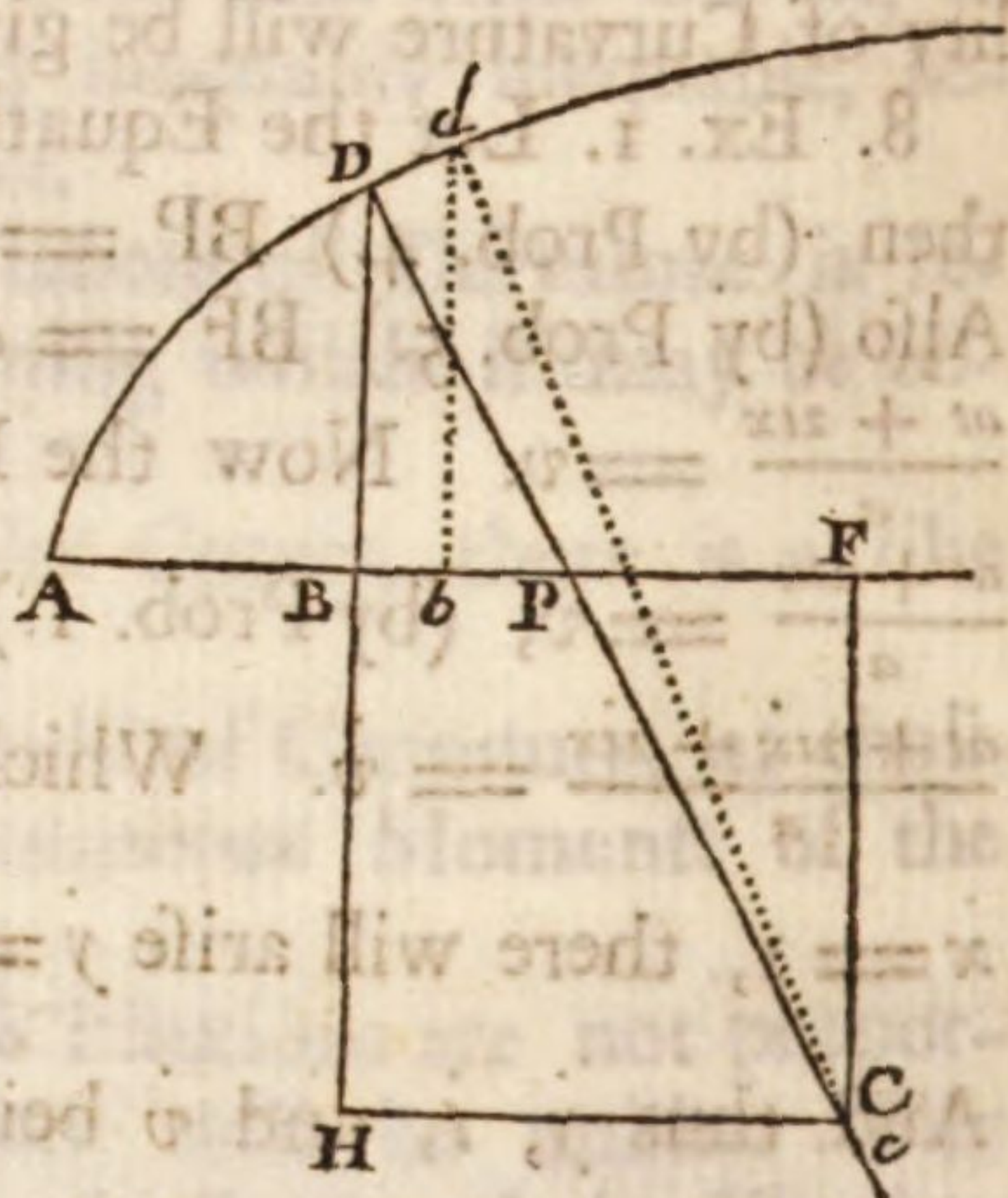
10. But if it were determin'd, that $x = 2$, then $y = 2$, $\dot{y} = \frac{1}{2}$, $t = \sqrt{5}$, $\dot{t} = \frac{1}{\sqrt{5}}$, and $\dot{v} = 3\sqrt{5}$. So that $\left(\frac{\dot{v}y}{t} = \right) 6$ will be here the Index of Inequability.

11. Wherefore the Inequability of Curvature at the Point of the Curve, from whence an Ordinate, equal to the Latus rectum of the Parabola, being drawn perpendicular to the Axis, will be double to the Inequability at that Point, from whence the Ordinate so drawn is half the Latus rectum; that is, the Curvature at the first Point is as unlike again to the Curvature of the Circle, as the Curvature at the second Point.

12. Ex. 2. Let the Equation be $2ax - bxx = yy$, and (by Prob. 4.) it will be $a - bx = BP$, and thence $tt = (aa - 2abx + b^2xx + yy =) aa - byy + yy$. Also (by Prob. 5.) it is $DH = y + \frac{y^3 - by^3}{aa}$, where, if for $yy - byy$ you substitute $tt - aa$, there arises $DH = \frac{ty}{aa}$. 'Tis also $BD : DP :: DH : DC = \frac{t^3}{a^2} = v$. Now (by Prob. 1.) the Equations $2ax - bxx = yy$, $aa - byy + yy = tt$, and $\frac{t^3}{aa} = v$, give

give $a - bx = yy$, and $yy - byy = tt$, and $\frac{3tt}{aa} = \dot{v}$. And thus $\dot{v}$ being found, the Index $\frac{\dot{v}y}{t}$ of the Inequability of Curvature, will also be known.

13. Thus in the Ellipsis $2x - 3xx = yy$, where it is $a = 1$, and $b = 3$; if we make $x = \frac{1}{2}$, then $y = \frac{1}{2}$, $\dot{y} = -1$, $t = \sqrt{\frac{1}{2}}$, $\dot{t} = \sqrt{2}$, $\dot{v} = 3\sqrt{\frac{1}{2}}$, and therefore $\frac{\dot{v}y}{t} = \frac{3}{2}$, which is the Index of the Inequability of Curvature. Hence it appears, that the Curvature of this Ellipsis, at the Point D here assign'd, is by two times less inequable, (or by two times more like to the Curvature of the Circle,) than the Curvature of the Parabola, at that Point of its Curve, from whence an Ordinate let fall upon the Axis is equal to half the Latus rectum.



14. If we have a mind to compare the Conclusions derived in these Examples, in the Parabola $2ax = yy$ arises $\left(\frac{\dot{v}y}{t}\right) \frac{3y}{a}$ for the Index of Inequability; and in the Ellipsis $2ax - bxx = yy$, arises $\left(\frac{\dot{v}y}{t}\right) \frac{3y - 3by}{aa} \times BP$; and so in the Hyperbola $2ax + bxx = yy$, the analogy being observed, there arises the Index $\left(\frac{\dot{v}y}{t}\right) \frac{3y + 3by}{aa} \times BP$. Whence it is evident, that at the different Points of any Conic Section consider'd apart, the Inequability of Curvature is as the Rectangle $BD \times BP$. And that, at the several Points of the Parabola, it is as the Ordinate BD .

15. Now as the Parabola is the most simple Figure of those that are curved with inequable Curvature, and as the Inequability of its Curvature is so easily determined, (for its Index is $6 \times \frac{\text{Ordinate}}{\text{Lat. rect.}}$), therefore the Curvatures of other Curves may not improperly be compared to the Curvature of this.

16. As if it were inquired, what may be the Curvature of the Ellipsis $2x - 3xx = yy$, at that Point of the Perimeter which is determined by assuming $x = \frac{1}{2}$: Because its Index is $\frac{3}{2}$, as before, it might be answer'd, that it is like the Curvature of the Parabola

$6x = yy$, at that Point of the Curve, between which and the Axis the perpendicular Ordinate is equal to $\frac{3}{4}$.

17. Thus, as the Fluxion of the Spiral ADE is to the Fluxion of the Subtense AD, in a certain given Ratio, suppose as d to e ; on its concave side erect

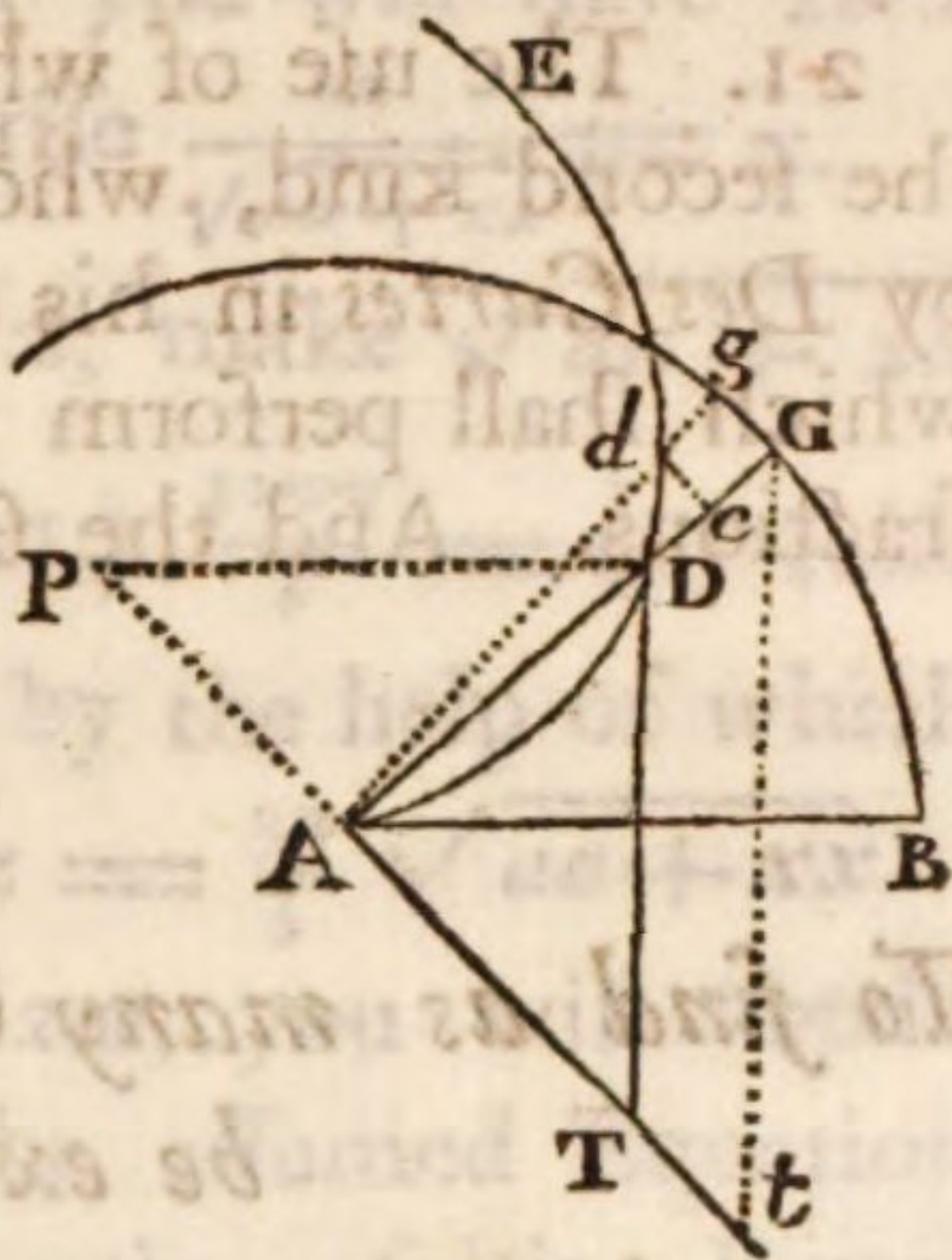
$AP = \frac{e}{\sqrt{dd - ee}} \times AD$ perpendicular to AD,

and P will be the Center of Curvature, and

$\frac{AP}{AD}$ or $\frac{e}{\sqrt{dd - ee}}$, will be the Index of Inequa-

bility. So that this Spiral has every where its Curvature alike inequable, as the Parabola $6x = yy$ has in that Point of its Curve, from whence to its Absciss a perpendicular Ordinate is let fall, which is equal to the

quantity $\frac{e}{\sqrt{dd - ee}}$.



18. And thus the Index of Inequability at any Point D of the Trochoid, (see Fig. in *Art. 29. pag. 64.*) is found to be $\frac{AB}{BL}$. Wherefore its Curvature at the same Point D is as inequable, or as unlike to that of a Circle, as the Curvature of any Parabola $ax = yy$ is at the Point where the Ordinate is $\frac{1}{2}a \times \frac{AB}{BL}$.

19. And from these Considerations the Sense of the Problem, as I conceive, must be plain enough; which being well understood, it will not be difficult for any one, who observes the Series of the things above deliver'd, to furnish himself with more Examples, and to contrive many other Methods of operation, as occasion may require. So that he will be able to manage Problems of a like nature, (where he is not discouraged by tedious and perplex Calculations,) with little or no difficulty. Such are these following;

I. To find the Point of any Curve, where there is either no Inequability of Curvature, or infinite, or the greatest, or the least.

20. Thus at the Vertices of the Conic Sections, there is no Inequability of Curvature; at the Cuspid of the Trochoid it is infinite; and it is greatest at those Points of the Ellipsis, where the Rectangle $BD \times BP$ is greatest, that is, where the Diagonal-Lines of the circumscribed Parallelogram cut the Ellipsis, whose Sides touch it in their principal Vertices.

II. To determine a Curve of some definite Species, suppose a Conic Section, whose Curvature at any Point may be equal and similar to the Curvature of any other Curve, at a given Point of it.

III.

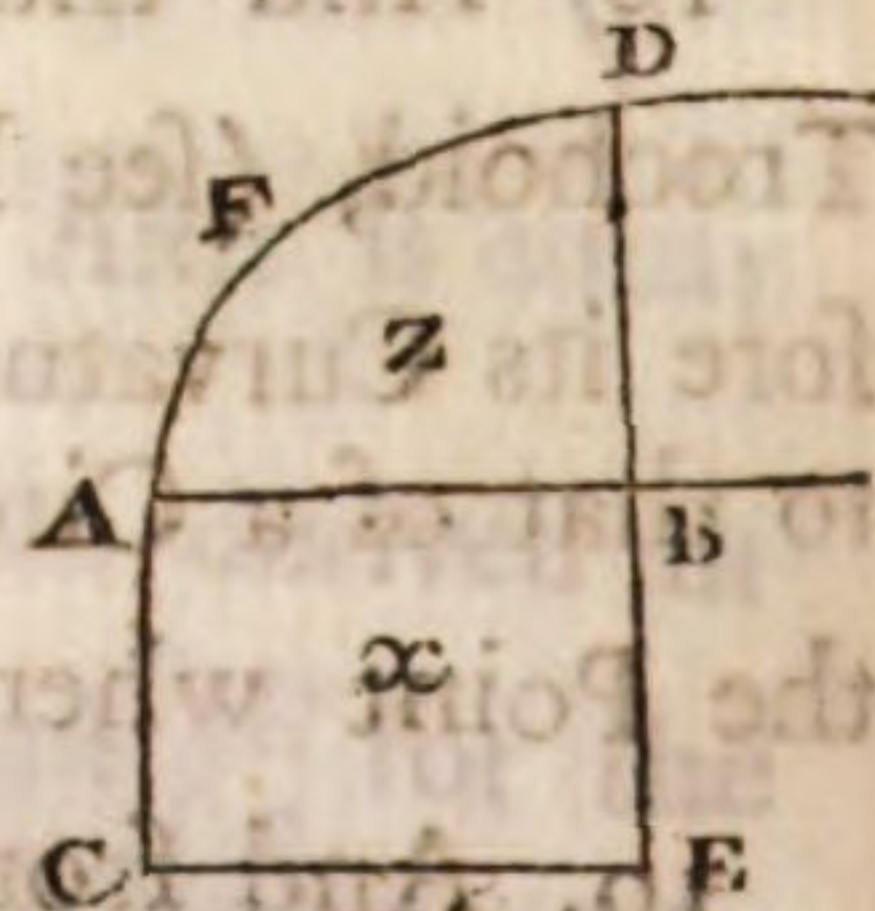
III. To determine a Conic Section, at any Point of which, the Curvature and Position of the Tangent, (in respect of the Axis,) may be like to the Curvature and Position of the Tangent, at a Point assign'd of any other Curve.

21. The use of which Problem is this, that instead of Ellipses of the second kind, whose Properties of refracting Light are explain'd by *Des Cartes* in his Geometry, Conic Sections may be substituted, which shall perform the same thing, very nearly, as to their Refractions. And the same may be understood of other Curves.

PROB. VII.

To find as many Curves as you please, whose Areas may be exhibited by finite Equations.

1. Let AB be the Absciss of a Curve, at whose Vertex A let the perpendicular AC = 1 be raised, and let CE be drawn parallel to AB. Let also DB be a rectangular Ordinate, meeting the right Line CE in E, and the Curve AD in D. And conceive these Areas ACEB and ADB to be generated by the right Lines BE and BD, as they move along the Line AB. Then their Increments or Fluxions will be always as the describing Lines BE and BD. Wherefore make the Parallelogram ACEB, or $AB \times 1 = x$, and the Area of the Curve ADB call z . And the Fluxions $\dot{x}$ and $\dot{z}$ will be as BE and BD; so that making $\dot{x} = 1 = BE$, then $\dot{z} = BD$.



2. Now if any Equation be assumed at pleasure, for determining the relation of z and x , from thence, (by Prob. 1.) may z be derived. And thus there will be two Equations, the latter of which will determine the Curve, and the former its Area.

EXAMPLES.

3. Assume $xx = z$, and thence (by Prob. 1.) $2\dot{x}x = \dot{z}$, or $2x = \dot{z}$, because $\dot{x} = 1$.

4. Assume $\frac{x^3}{a} = z$, and thence will arise $\frac{3x^2}{a} = \dot{z}$, an Equation to the Parabola.

5. Assume $ax^3 = zz$, or $a^{\frac{1}{2}}x^{\frac{3}{2}} = z$, and there will arise $\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}} = \dot{z}$, or $\frac{3}{4}ax = z\dot{z}$, an Equation again to the Parabola.

6. Assume $a^6 x^{-2} = zx$, or $a^3 x^{-1} = z$, and there arises $-a^3 x^{-2} = \dot{z}$, or $a^3 + zxx = 0$. Here the negative Value of $\dot{z}$ only insinuates, that BD is to be taken the contrary way from BE.

7. Again if you assume $c^2 a^2 + c^2 x^2 = z^2$, you will have $2c^2 x = 2z\dot{z}$; and z being eliminated, there will arise $\frac{cx}{\sqrt{aa+xx}} = \dot{z}$.

8. Or if you assume $\frac{aa+xx}{b} \sqrt{aa+xx} = z$, make $\sqrt{aa+xx} = v$, and it will be $\frac{v^3}{b} = z$, and then (by Prob. I.) $\frac{3v\dot{v}}{b} = \dot{z}$. Also the Equation $aa+xx = vv$ gives $2x = 2v\dot{v}$, by the help of which if you exterminate $\dot{v}$, it will become $\frac{3vx}{b} = \dot{z} = \frac{3x}{b} \sqrt{aa+xx}$.

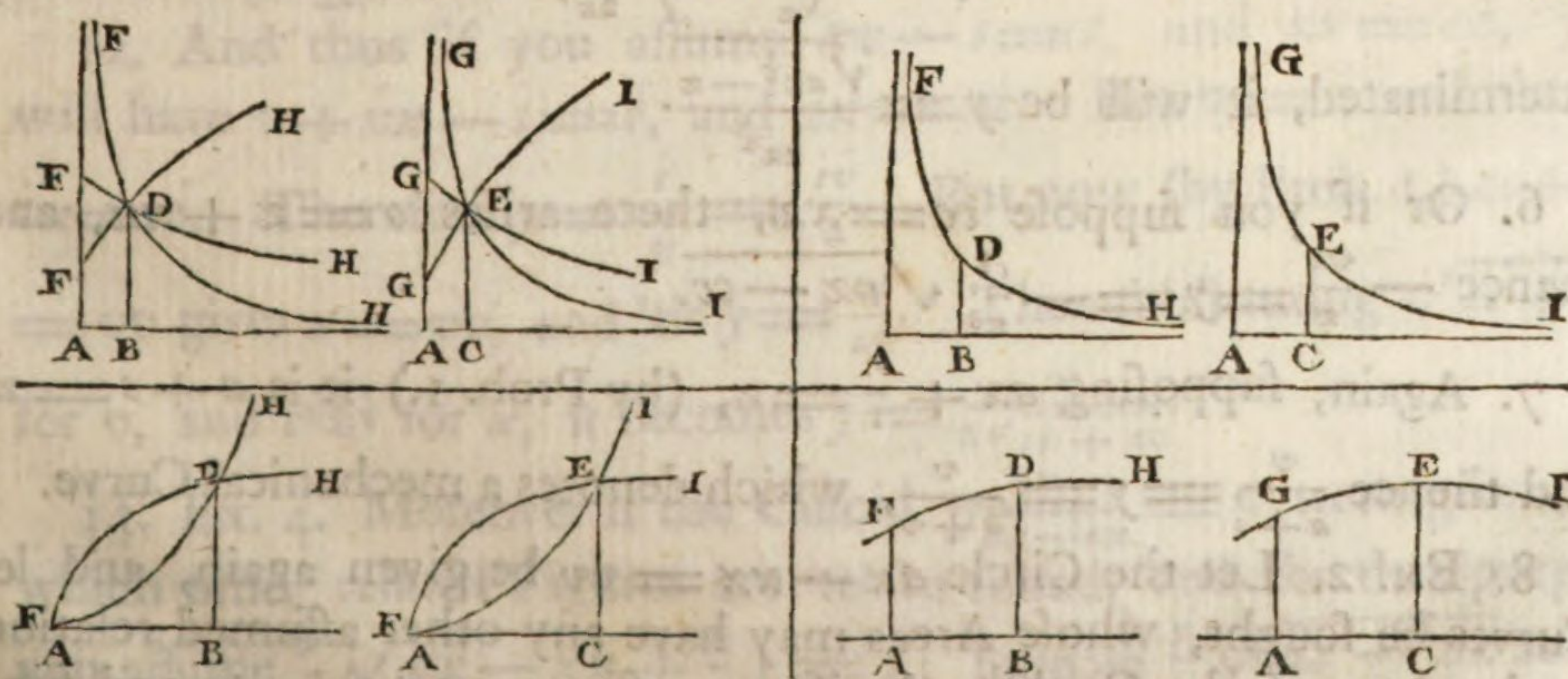
9. Lastly, if you assume $8 - 3xz + \frac{1}{3}z = zx$, you will obtain $-3z - 3xz + \frac{1}{3}z = 2z\dot{z}$. Wherefore by the assumed Equation first seek the Area z , and then the Ordinate $\dot{z}$ by the resulting Equation.

10. And thus from the Areas, however they may be feign'd, you may always determine the Ordinates to which they belong.

PROB. VIII.

To find as many Curves as you please, whose Areas shall have a relation to the Area of any given Curve, assignable by finite Equations.

1. Let FDH be a given Curve, and GEI the Curve required, and conceive their Ordinates DB and EC to move at right Angles upon



their Abscisses or Bases AB and AC. Then the Increments or Fluxions of the Areas which they describe, will be as those Ordinates drawn
M into

into their Velocities of moving, that is, into the Fluxions of their Abscisses. Therefore make $AB = x$, $BD = v$, $AC = z$, and $CE = y$, the Area $AFDB = s$, and the Area $AGEC = t$, and let the Fluxions of the Areas be $\dot{s}$ and $\dot{t}$: And it will be $\dot{x}v : \dot{z}y :: \dot{s} : \dot{t}$. Therefore if we suppose $\dot{x} = 1$, and $v = s$, as before; it will be $\dot{z}y = \dot{t}$, and thence $\frac{\dot{t}}{z} = y$.

2. Therefore let any two Equations be assumed; one of which may express the relation of the Areas s and t , and the other the relation of their Abscisses x and z , and thence, (by Prob. 1.) let the Fluxions $\dot{t}$ and $\dot{z}$ be found, and then make $\frac{\dot{t}}{z} = y$.

3. Ex. 1. Let the given Curve FDH be a Circle, express'd by the Equation $ax - xx = vv$, and let other Curves be sought, whose Areas may be equal to that of the Circle. Therefore by the Hypothesis $s = t$, and thence $\dot{s} = \dot{t}$, and $y = \frac{\dot{t}}{z} = \frac{v}{z}$. It remains to determine $\dot{z}$, by assuming some relation between the Abscisses x and z .

4. As if you suppose $ax = zz$; then (by Prob. 1.) $a = 2z\dot{z}$: So that substituting $\frac{a}{2z}$ for $\dot{z}$, then $y = \frac{v}{z} = \frac{2vz}{a}$. But it is $v = (\sqrt{ax - xx}) = \frac{z}{a} \sqrt{aa - zz}$, therefore $\frac{2zz}{aa} \sqrt{aa - zz} = y$ is the Equation to the Curve, whose Area is equal to that of the Circle.

5. After the same manner if you suppose $xx = z$, there will arise $2x = \dot{z}$, and thence $y = \left(\frac{v}{z}\right) \frac{v}{2x}$; whence v and x being exterminated, it will be $y = \frac{\sqrt{ax^{\frac{1}{2}} - z}}{2z^{\frac{1}{2}}}$.

6. Or if you suppose $cc = xz$, there arises $c = z + x\dot{z}$, and thence $-\frac{vx}{z} = y = -\frac{c^3}{z^3} \sqrt{az - cc}$.

7. Again, supposing $ax + \frac{s}{1} = z$, (by Prob. 1.) it is $a + \dot{s} = \dot{z}$, and thence $\frac{v}{a + \dot{s}} = y = \frac{v}{a + v}$, which denotes a mechanical Curve.

8. Ex. 2. Let the Circle $ax - xx = vv$ be given again, and let Curves be sought, whose Areas may have any other assumed relation to the Area of the Circle. As if you assume $cx + s = t$, and suppose also $ax = zz$. (By Prob. 1.) 'tis $c + \dot{s} = \dot{t}$, and $a = 2z\dot{z}$.
I Therefore

Therefore $y = \frac{\dot{t}}{z} = \frac{2cz + 2sz}{a}$; and substituting $\sqrt{ax - xx}$ for s , and $\frac{zx}{a}$ for x , 'tis $y = \frac{2cz}{a} + \frac{2zx}{aa} \sqrt{aa - zz}$.

9. But if you assume $s - \frac{2v^3}{3a} = t$, and $x = z$, you will have $\dot{s} - \frac{2\dot{v}v^2}{a} = \dot{t}$, and $1 = z$. Therefore $y = \frac{\dot{t}}{z} = \dot{s} - \frac{2\dot{v}v^2}{a}$, or $= v - \frac{2\dot{v}v^2}{a}$. Now for exterminating $\dot{v}$, the Equation $ax - xx = vv$, (by Prob. 1.) gives $a - 2x = 2\dot{v}v$, and therefore 'tis $y = \frac{2vx}{a}$. Where if you expunge v and x by substituting their values $\sqrt{ax - xx}$ and z , there will arise $y = \frac{2z}{a} \sqrt{az - zz}$.

10. But if you assume $ss = t$, and $x = zz$, there will arise $2s\dot{s} = \dot{t}$, and $1 = 2z\dot{z}$; and therefore $y = \frac{\dot{t}}{z} = 4s\dot{s}z$. And for s and x substituting $\sqrt{ax - xx}$ and zz , it will become $y = 4sz\dot{z} \sqrt{a - zz}$, which is an Equation to a mechanical Curve.

11. Ex. 3. After the same manner Figures may be found, which have an assumed relation to any other given Figure. Let the Hyperbola $cc + xx = vv$ be given; then if you assume $s = t$, and $xx = cz$, you will have $\dot{s} = \dot{t}$ and $2x = c\dot{z}$; and thence $y = \frac{\dot{t}}{z} = \frac{c\dot{z}}{2z}$. Then substituting $\sqrt{cc + xx}$ for s , and $c^{\frac{1}{2}}z^{\frac{1}{2}}$ for x , it will be $y = \frac{c}{2z} \sqrt{cz + zz}$.

12. And thus if you assume $xv - s = t$, and $xx = cz$, you will have $v + vx - s = t$, and $2x = c\dot{z}$. But $v = s$, and thence $\dot{v}x = \dot{t}$. Therefore $y = \frac{\dot{t}}{z} = \frac{c\dot{v}}{z}$. But now (by Prob. 1.) $cc + xx = vv$ gives $x = vv$, and 'tis $y = \frac{cx}{2v}$. Then substituting $\sqrt{cc + xx}$ for v , and $c^{\frac{1}{2}}z^{\frac{1}{2}}$ for x , it becomes $y = \frac{cz}{2\sqrt{cz + zz}}$.

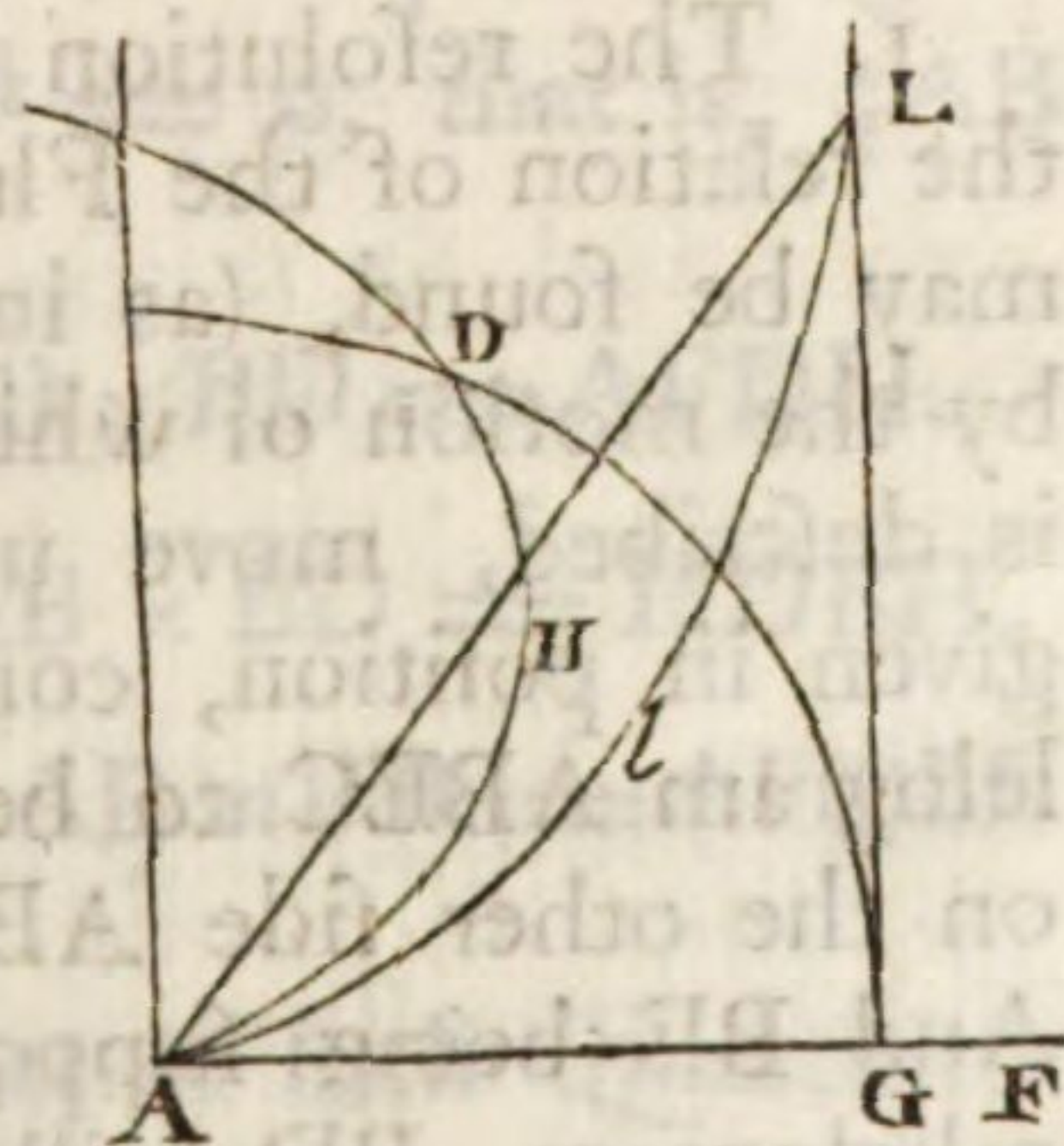
13. Ex. 4. Moreover if the Cissoïd $\frac{xx}{\sqrt{ax - xx}} = v$ were given, to which other related Figures are to be found, and for that purpose you assume $\frac{x}{3} \sqrt{ax - xx} + \frac{2}{3}s = t$; suppose $\frac{x}{3} \sqrt{ax - xx} = b$, and its Fluxion $\dot{b}$; therefore $\dot{b} + \frac{2}{3}\dot{s} = \dot{t}$. But the Equation $\frac{ax^3 - x^4}{9} = bb$

that is, as 1 to $\dot{v}$. So that $\dot{v} = \frac{BL}{AB}$, and therefore $x\dot{v} = BL$. Therefore the Area ADG is described by the Fluxion BL; since therefore the circular Area ALB is described by the same Fluxion, they will be equal.

17. In like manner if you conceive ADF to be a Figure of Arches, or of versed Sines, that is, whose Ordinate BD is equal to the Arch AL; since the Fluxion of the Arch AL is to the Fluxion of the Absciss AB, as PL to BL, that is, $\dot{v} : 1 :: \frac{1}{2}a : \sqrt{ax - xx}$, then $\dot{v} = \frac{a}{2\sqrt{ax - xx}}$. Then $\dot{v}x$, the Fluxion of the Area ADG, will be $\frac{ax}{2\sqrt{ax - xx}}$. Wherefore if a right Line equal to $\frac{ax}{2\sqrt{ax - xx}}$ be conceived to be apply'd as a rectangular Ordinate at B, a point of the Line AB, it will be terminated at a certain geometrical Curve, whose Area, adjoining to the Absciss AB, is equal to the Area ADG.

18. And thus geometrical Figures may be found equal to other Figures, made by the application (in any Angle) of Arches of a Circle, of an Hyperbola, or of any other Curve, to the Sines right or versed of those Arches, or to any other right Lines that may be Geometrically determin'd.

19. As to Spirals, the matter will be very short. For from the Center of Rotation A, the Arch DG being described, with any Radius AG, cutting the right Line AF in G, and the Spiral in D; since that Arch, as a Line moving upon the Absciss AG, describes the Area of the Spiral AHDG, so that the Fluxion of that Area is to the Fluxion of the Rectangle $1 \times AG$, as the Arch GD to 1; if you raise the perpendicular right Line GL equal to that Arch, by moving in like manner upon the same Line AG, it will describe the Area A/LG equal to the Area of the Spiral AHDG: The Curve A/L being a geometrical Curve.



And further, if the Subtense AL be drawn, then $\triangle ALG = \frac{1}{2}AG \times GL = \frac{1}{2}AG \times GD = \text{Sector AGD}$; therefore the complementary Segments ALl and ADH will also be equal. And this not only agrees to the Spiral of *Archimedes*, (in which case A/L becomes the Parabola of *Apollonius*;) but to any other whatever; so that all of them may be converted into equal geometrical Curves with the same ease.

20. I might have produced more Specimens of the Construction of this Problem, but these may suffice; as being so general, that whatever as yet has been found out concerning the Areas of Curves, or (I believe) can be found out, is in some manner contain'd herein, and is here determined for the most part with less trouble, and without the usual perplexities.

21. But the chief use of this and the foregoing Problem is, that assuming the Conic Sections, or any other Curves of a known magnitude, other Curves may be found out that may be compared with these, and that their defining Equations may be disposed orderly in a Catalogue or Table. And when such a Table is constructed, when the Area of any Curve is to be found, if its defining Equation be either immediately found in the Table, or may be transformed into another that is contain'd in the Table, then its Area may be known. Moreover such a Catalogue or Table may be apply'd to the determining of the Lengths of Curves, to the finding of their Centers of Gravity, their Solids generated by their rotation, the Superficies of those Solids, and to the finding of any other flowing quantity produced by a Fluxion analogous to it.

P R O B. IX.

To determine the Area of any Curve proposed.

1. The resolution of the Problem depends upon this, that from the relation of the Fluxions being given, the relation of the Fluents may be found, (as in Prob. 2.) And first, if the right Line BD, by the motion of which the Area required AFDB is described, move upright upon an Absciss AB given in position, conceive (as before) the Parallelogram ABEC to be described in the mean time on the other side AB, by a line equal to unity. And BE being suppos'd the Fluxion of the Parallelogram, BD will be the Fluxion of the Area required.



2. Therefore make $AB = x$, and then also $ABEC = 1 \times x = x$, and $BE = \dot{x}$. Call also the Area $AFDB = z$, and it will be $BD = \dot{z}$, as also $= \frac{\dot{z}}{x}$, because $x = 1$. Therefore by the Equation expressing BD, at the same time the ratio of the Fluents $\frac{z}{x}$

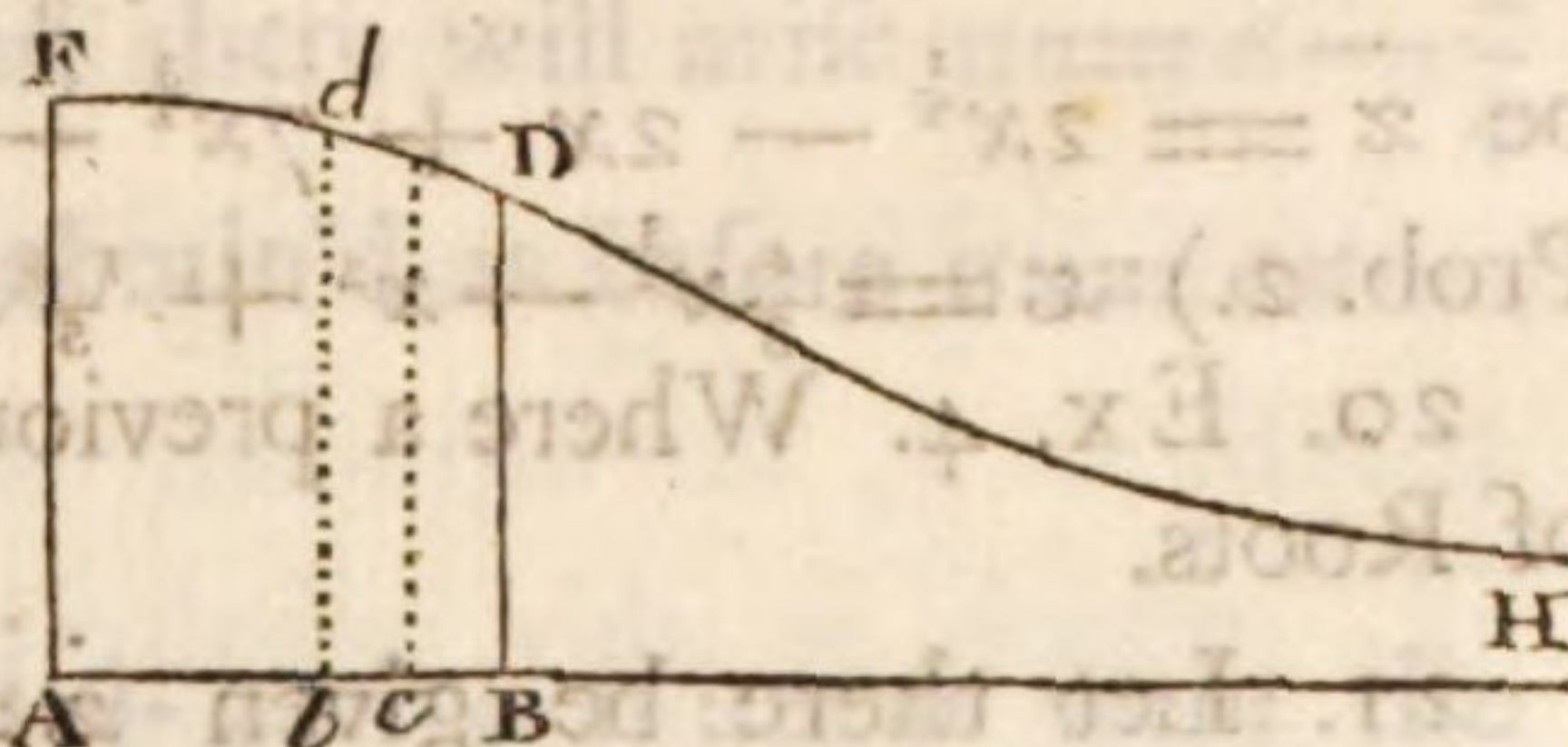
is express'd, and thence (by Prob. 2. Case 1.) may be found the relation of the flowing quantities x and z .

3. Ex. 1. When BD, or z , is equal to some simple quantity.

4. Let there be given $\frac{xx}{a} = z$, or $\frac{z}{x}$, (the Equation to the Parabola,) and (Prob. 2.) there will arise $\frac{x^3}{3a} = z$. Therefore $\frac{x^3}{3a}$, or $\frac{1}{3} AB \times BD$, = Area of the Parabola AFDB.

5. Let there be given $\frac{x^3}{aa} = z$, (an Equation to a Parabola of the second kind,) and there will arise $\frac{x^4}{4a^2} = z$, that is, $\frac{1}{4} AB \times BD$ = Area AFDB.

6. Let there be given $\frac{a^3}{xx} = z$, or $a^3 x^{-2} = z$, (an Equation to an Hyperbola of the second kind,) and there will arise $-a^3 x^{-1} = z$, or $-\frac{a^3}{x} = z$. That is, $AB \times BD$ = Area HDBH, of an infinite length, lying on the other side of the Ordinate BD, as its negative value insinuates.



7. And thus if there were given $\frac{a^4}{x^3} = z$, there would arise $-\frac{a^4}{2xx} = z$.

8. Moreover, let $ax = zz$, or $a^{\frac{1}{2}} x^{\frac{1}{2}} = z$, (an Equation again to the Parabola,) and there will arise $\frac{2}{3} a^{\frac{1}{2}} x^{\frac{3}{2}} = z$, that is, $\frac{2}{3} AB \times BD$ = Area AFDB.

9. Let $\frac{a^3}{x} = zz$; then $-2a^{\frac{3}{2}} x^{\frac{1}{2}} = z$, or $2 AB \times BD$ = AFDH.

10. Let $\frac{a^5}{x^3} = zz$; then $-\frac{2a^{\frac{5}{2}}}{x^{\frac{1}{2}}} = z$, or $2 AB \times BD$ = HDBH.

11. Let $ax^2 = z^3$; then $\frac{3}{2} a^{\frac{1}{3}} x^{\frac{5}{3}} = z$, or $\frac{3}{2} AB \times BD$ = AFDH. And so in others.

12. Ex. 2. Where z is equal to an Aggregate of such Quantities.

13. Let $x + \frac{xx}{a} = z$; then $\frac{xx}{2} + \frac{xxx}{3a} = z$.

14. Let $a + \frac{a^3}{xx} = z$; then $ax - \frac{a^3}{x} = z$.

15. Let $3x^{\frac{1}{2}} - \frac{5}{xx} - \frac{2}{x^{\frac{1}{2}}} = z$; then $2x^{\frac{3}{2}} + \frac{5}{x} - 4x^{\frac{1}{2}} = z$.

16. Ex. 3. Where a previous reduction by Division is required.

17. Let there be given $\frac{aa}{b+x} = z$ (an Equation to the Apollonian Hyperbola,) and the division being performed *in infinitum*, it will be

$\dot{z} = \frac{aa}{b} - \frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{aax^3}{b^4}, \&c.$ And thence, (by Prob. 2.) as in the second Set of Examples, you will obtain $z = \frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{a^2x^4}{4b^4}, \&c.$

18. Let there be given $\frac{1}{1+xx} = \dot{z}$, and by division it will be $\dot{z} = 1 - x^2 + x^4 - x^6, \&c.$ or else $\dot{z} = \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6}, \&c.$ And thence (by Prob. 2.) $z = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7, \&c. = \text{AFDB};$ or $z = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}, \&c. = \text{HDBH}.$

19. Let there be given $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = \dot{z}$, and by division it will be $\dot{z} = 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}, \&c.$ And thence (by Prob. 2.) $z = \frac{4}{3}x^{\frac{3}{2}} - x^2 + \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{3}x^3 + \frac{6}{7}x^{\frac{7}{2}}, \&c.$

20. Ex. 4. Where a previous reduction is required by Extraction of Roots.

21. Let there be given $\dot{z} = \sqrt{aa + xx}$, (an Equation to the Hyperbola,) and the Root being extracted to an infinite multitude of terms, it will be $\dot{z} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{112a^7}, \&c.$ whence as in the foregoing $z = ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7}, \&c.$

22. In the same manner if the Equation $\dot{z} = \sqrt{aa - xx}$ were given, (which is to the Circle,) there would be produced $z = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7}, \&c.$

23. And so if there were given $\dot{z} = \sqrt{x - xx}$, (an Equation also to the Circle,) by extracting the Root there would arise $\dot{z} = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}, \&c.$ And therefore $z = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}, \&c.$

24. Thus $\dot{z} = \sqrt{aa + bx - xx}$, (an Equation again to the Circle,) by extraction of the Root it gives $\dot{z} = a + \frac{bx}{2a} - \frac{xx}{2a} - \frac{b^2x^2}{8a^3}, \&c.$ whence $z = ax + \frac{bx^2}{4a} - \frac{x^3}{6a} - \frac{b^2x^3}{24a^3}, \&c.$

25. And thus $\sqrt{\frac{1+axx}{1-bxx}} = \dot{z}$, by a due reduction gives $\dot{z} = 1 + \frac{1}{2}bx^2 + \frac{3}{8}bbx^4, \&c.$ then $z = x + \frac{1}{6}bx^3 + \frac{3}{40}bbx^5, \&c.$
 $\quad \quad \quad + \frac{1}{2}a \quad + \frac{1}{4}ab \quad \quad \quad + \frac{1}{6}a \quad + \frac{1}{20}ab$
 $\quad \quad \quad - \frac{1}{8}aa \quad \quad \quad - \frac{1}{40}aa$

26. Thus finally $z = \sqrt[3]{a^3 + x^3}$, by the extraction of the Cubic Root, gives $z = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8}$, &c. and then (by Prob. 2.) $z = ax + \frac{x^4}{12a^2} - \frac{x^7}{63a^5} + \frac{x^{10}}{162a^8}$, &c. = AFDB. Or else $z = x + \frac{a^3}{3xx} - \frac{a^6}{9x^5} + \frac{5a^9}{81x^8}$, &c. And thence $z = \frac{x^2}{2} - \frac{a^3}{3x} + \frac{a^6}{36x^4} - \frac{5a^9}{567x^7}$, &c. = HDBH.

27. Ex. 5. Where a previous reduction is required, by the resolution of an affected Equation.

28. If a Curve be defined by this Equation $z^3 + a^2z + axz - 2a^3 - x^3 = 0$, extract the Root, and there will arise $z = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^4}{512aa}$, &c. whence will be obtain'd as before $z = ax - \frac{xx}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}$, &c.

29. But if $z^3 - cz^2 - 2x^2z - c^2z + 2x^3 + c^3 = 0$ were the Equation to the Curve, the resolution will afford a three-fold Root; either $z = c + x - \frac{xx}{4c} + \frac{x^3}{32c^2}$, &c. or $z = c - x + \frac{3x^2}{4c} - \frac{15x^3}{32cc}$, &c. or $z = -c - \frac{x^2}{2c} - \frac{x^3}{2cc} + \frac{x^5}{4c^4}$, &c. And hence will arise the values of the three corresponding Areas, $z = cx + \frac{1}{2}x^2 - \frac{x^3}{12c} + \frac{x^4}{128c^2}$, &c. $z = cx - \frac{1}{2}x^2 + \frac{x^3}{4c} - \frac{15x^4}{128c^2}$, &c. and $z = -cx - \frac{x^3}{6c} - \frac{x^4}{8c^2} + \frac{x^6}{24c^4}$, &c.

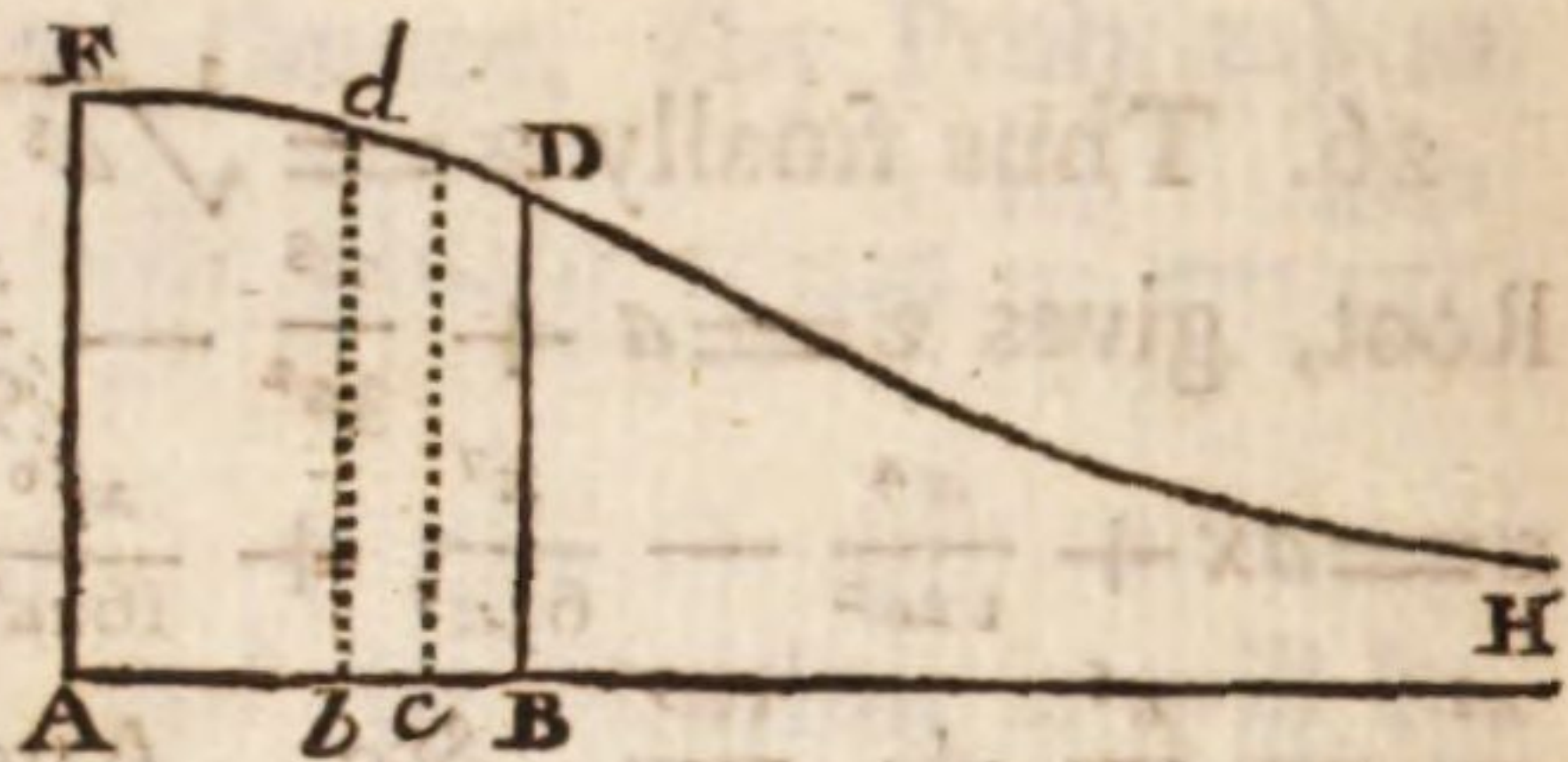
30. I add nothing here concerning mechanical Curves, because their reduction to the form of geometrical Curves will be taught afterwards.

31. But whereas the values of z thus found belong to Areas which are situate, sometimes to a finite part AB of the Absciss, sometimes to a part BH produced infinitely towards H, and sometimes to both parts, according to their different terms: That the due value of the Area may be assign'd, adjacent to any portion of the Absciss, that Area is always to be made equal to the difference of the values of z , which belong to the parts of the Absciss, that are terminated at the beginning and end of the Area.

32. For Instance; to the Curve express'd by the Equation $\frac{1}{1+xx} = z$,

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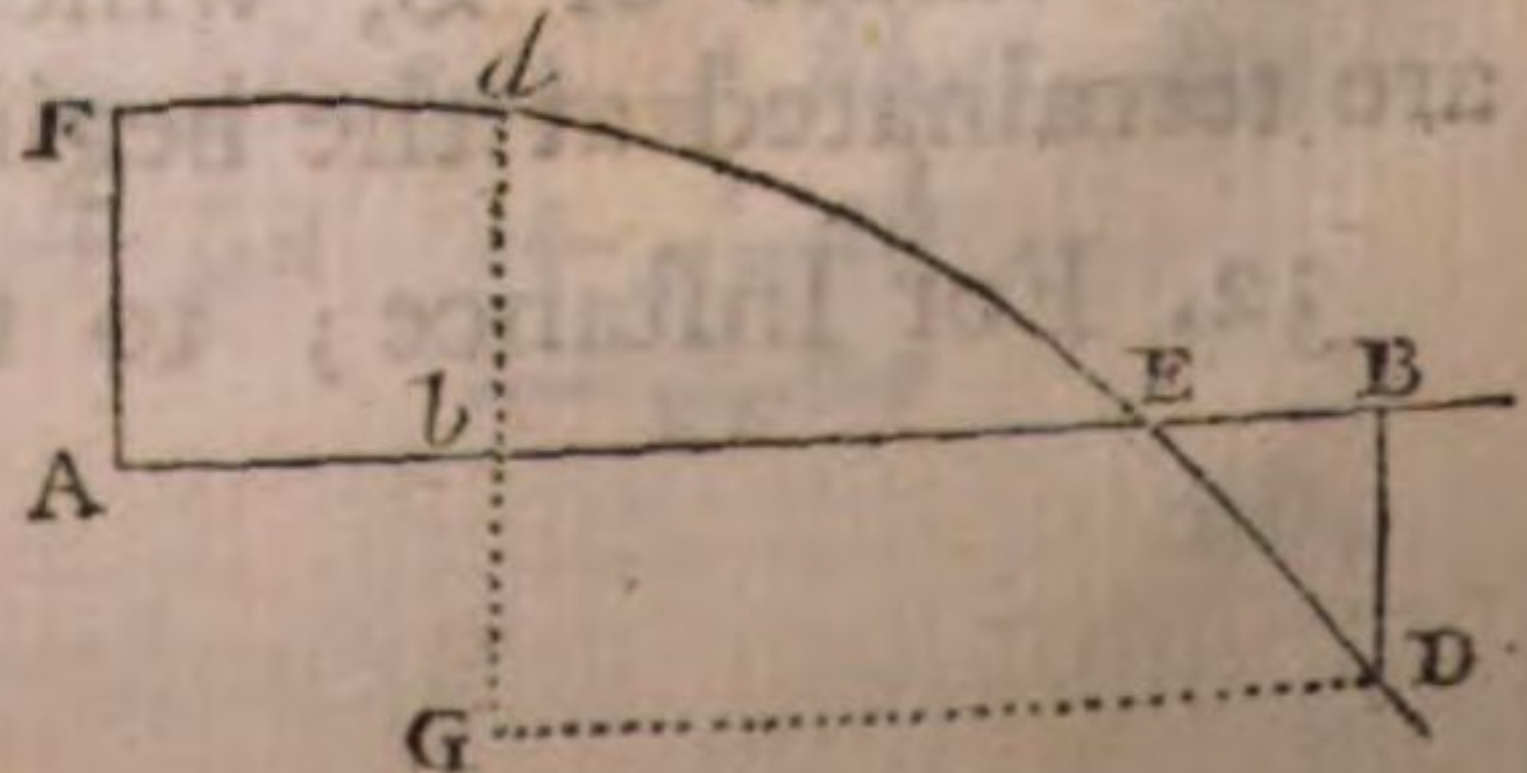
$=z$, it is found that $z = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c. Now that I may determine the quantity of the Area $bdDB$, adjacent to the part of the Absciss bB ; from the value of z which arises by putting $AB = x$, I take the value of z which arises by putting $Ab = x$, and there remains $x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c. $-x + \frac{1}{3}x^3 - \frac{1}{5}x^5$, &c. the value of that Area $bdDB$. Whence if Ab , or x , be put equal to nothing, there will be had the whole Area $AFDB = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$, &c.



33. To the same Curve there is also found $z = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. Whence again, according to what is before, the Area $bdDB = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5}$, &c. $-\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. Therefore if AB , or x , be supposed infinite, the adjoining Area bdH toward H , which is also infinitely long, will be equivalent to $\frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5}$, &c. For the latter Series $-\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$, &c. will vanish, because of its infinite denominators.

34. To the Curve represented by the Equation $a + \frac{a^3}{xx} = z$, it is found, that $z = ax - \frac{a^3}{x}$. Whence it is that $ax - \frac{a^3}{x} = ax + \frac{a^3}{x} = \text{Area } bdDB$. But this becomes infinite, whether x be supposed nothing, or x infinite; and therefore each Area $AFDB$ and bdH is infinitely great, and the intermediate parts alone, such as $bdDB$, can be exhibited. And this always happens when the Absciss x is found as well in the numerators of some of the terms, as in the denominators of others, of the value of z . But when x is only found in the numerators, as in the first Example, the value of z belongs to the Area situate at AB , on this side the Ordinate. And when it is only in the denominators, as in the second Example, that value, when the signs of all the terms are changed, belongs to the whole Area infinitely produced beyond the Ordinate.

35. If at any time the Curve-line cuts the Absciss, between the points b and B , suppose in E , instead of the Area will be had the difference $bdE - BDE$ of the Areas at the different parts of the Absciss; to which if there be added the Rectangle $BDGb$, the Area $dEDG$ will be obtain'd.



36. But it is chiefly to be regarded, that when in the value of z any term is divided by x of only one dimension; the Area corresponding to that term belongs to the Conical Hyperbola; and therefore is to be exhibited by it self, in an infinite Series: As is done in what follows.

37. Let $\frac{a^3 - a^2x}{ax + xx} = z$ be an Equation to a Curve; and by division it becomes $z = \frac{aa}{x} - 2a + 2x - \frac{2x^2}{a} + \frac{2x^3}{aa}, \&c.$ and thence $z = \left[\frac{aa}{x} \right] - 2ax + x^2 - \frac{2x^3}{3a} + \frac{x^4}{2a^2}, \&c.$ And the Area $bdDB = \left[\frac{aa}{x} \right] - 2ax + x^2 - \frac{2x^3}{3a}, \&c. - \left[\frac{aa}{x} \right] + 2ax - xx + \frac{2x^3}{3a}, \&c.$

Where by the Marks $\left[\frac{aa}{x} \right]$ and $\left[\frac{aa}{x} \right]$ I denote the little Areas belonging to the Terms $\frac{aa}{x}$ and $\frac{aa}{x}$.

38. Now that $\left[\frac{aa}{x} \right]$ and $\left[\frac{aa}{x} \right]$ may be found, I make Ab , or x , to be definite, and bB indefinite, or a flowing Line, which therefore I call y ; so that it will be $\left[\frac{aa}{x+y} \right] =$ to that Hyperbolical Area adjoining to bB , that is, $\left[\frac{aa}{x} \right] - \left[\frac{aa}{x} \right]$. But by Division it will be $\frac{aa}{x+y} = \frac{aa}{x} - \frac{a^2y}{x^2} + \frac{a^2y^2}{x^3} - \frac{a^2y^3}{x^4}, \&c.$ and therefore, $\left[\frac{aa}{x+y} \right]$ or $\left[\frac{aa}{x} \right] - \left[\frac{aa}{x} \right] = \frac{a^2y}{x} - \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3} - \frac{a^2y^4}{4x^4}, \&c.$ and therefore the whole Area required $bdDB = \frac{a^2y}{x} - \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3}, \&c. - 2ax + x^2 - \frac{2x^3}{3a}, \&c. + 2ax - xx + \frac{2x^3}{3a}, \&c.$

39. After the same manner, AB , or x , might have been used for a definite Line, and then it would have been $\left[\frac{aa}{x} \right] - \left[\frac{aa}{x} \right] = \frac{a^2y^2}{x} + \frac{a^2y^2}{2x^2} + \frac{a^2y^3}{3x^3} + \frac{a^2y^4}{4x^4}, \&c.$

40. Moreover, if bB be bisected in C , and AC be assumed to be of a definite length, and Cb and CB indefinite; then making $AC = e$, and Cb or $CB = y$, 'twill be $bd = \frac{aa}{e-y} = \frac{aa}{e} + \frac{a^2y}{e^2} + \frac{a^2y^2}{e^3} + \frac{a^2y^3}{e^4} + \frac{a^2y^4}{e^5}, \&c.$ and therefore the Hyperbolical Area adjacent to

to the Part of the Absciss bC will be $\frac{a^2y}{e} + \frac{a^2y^2}{2e^2} + \frac{a^2y^3}{3e^3} + \frac{a^2y^4}{4e^4}$,
 &c. 'Twill be also $DB = \frac{aa}{e+y} = \frac{aa}{e} - \frac{aay}{e^2} + \frac{aay^2}{e^3} - \frac{aa^3}{e^4} + \frac{aay^4}{e^5}$,
 &c. And therefore the Area adjacent to the other part of the Absciss CB
 $= \frac{a^2y}{e} - \frac{a^2y^2}{2e^2} + \frac{a^2y^3}{3e^3} - \frac{a^2y^4}{4e^4} + \frac{a^2y^5}{5e^5}$, &c. And the Sum of these
 Areas $\frac{2a^2y}{e} + \frac{2a^2y^3}{3e^3} - \frac{2a^2y^5}{5e^5}$, &c. will be equivalent to $\left[\frac{aa}{x} \right]$
 $- \left[\frac{aa}{x} \right]$.

41. Thus in the Equation $z^3 + z^2 + z - x^3 = 0$, denoting the
 nature of a Curve, its Root will be $z = x - \frac{1}{3} - \frac{2}{9x} + \frac{7}{81xx} + \frac{5}{81x^3}$,
 &c. Whence there arises $z = \frac{1}{2}xx - \frac{1}{3}x - \left[\frac{2}{9x} \right] - \frac{7}{81x} - \frac{5}{162xx}$,
 &c. And the Area $bdDB = \frac{1}{2}x^2 - \frac{1}{3}x - \left[\frac{2}{9x} \right] - \frac{7}{81x}$, &c.
 $- \frac{1}{2}xx + \frac{1}{3}x + \left[\frac{2}{9x} \right] + \frac{7}{81x}$, &c. that is, $= \frac{1}{2}x^2 - \frac{1}{3}x - \frac{7}{81x}$
 &c. $- \frac{1}{2}x^2 + \frac{1}{3}x + \frac{7}{81x}$, &c. $- \frac{4y}{9e} - \frac{4y^3}{27e^3} - \frac{4y^5}{45e^5}$, &c.

42. But this Hyperbolical term, for the most part, may be very
 commodiously avoided, by altering the beginning of the Absciss,
 that is, by increasing or diminishing it by some given quantity. As
 in the former Example, where $\frac{a^3 - a^2x}{ax + xx} = z$ was the Equation to
 the Curve, if I should make b to be the beginning of the Absciss,
 and supposing Ab to be of any determinate length $\frac{1}{2}a$, for the re-
 mainder of the Absciss bB , I shall now write x : That is, if I dimi-
 nish the Absciss by $\frac{1}{2}a$, by writing $x + \frac{1}{2}a$ instead of x , it will
 become $\frac{\frac{1}{2}a^3 - a^2x}{\frac{3}{4}a^2 + 2ax + x^2} = z$, and (by Division) $z = \frac{2}{3}a - \frac{2}{9}x$
 $+ \frac{200x^2}{27a}$, &c. whence arises $z = \frac{2}{3}ax - \frac{1}{9}x^2 + \frac{200x^3}{81a}$, &c. =
 Area $bdDB$.

43. And thus by assuming another and another point for the be-
 ginning of the Absciss, the Area of any Curve may be express'd an
 infinite variety of ways.

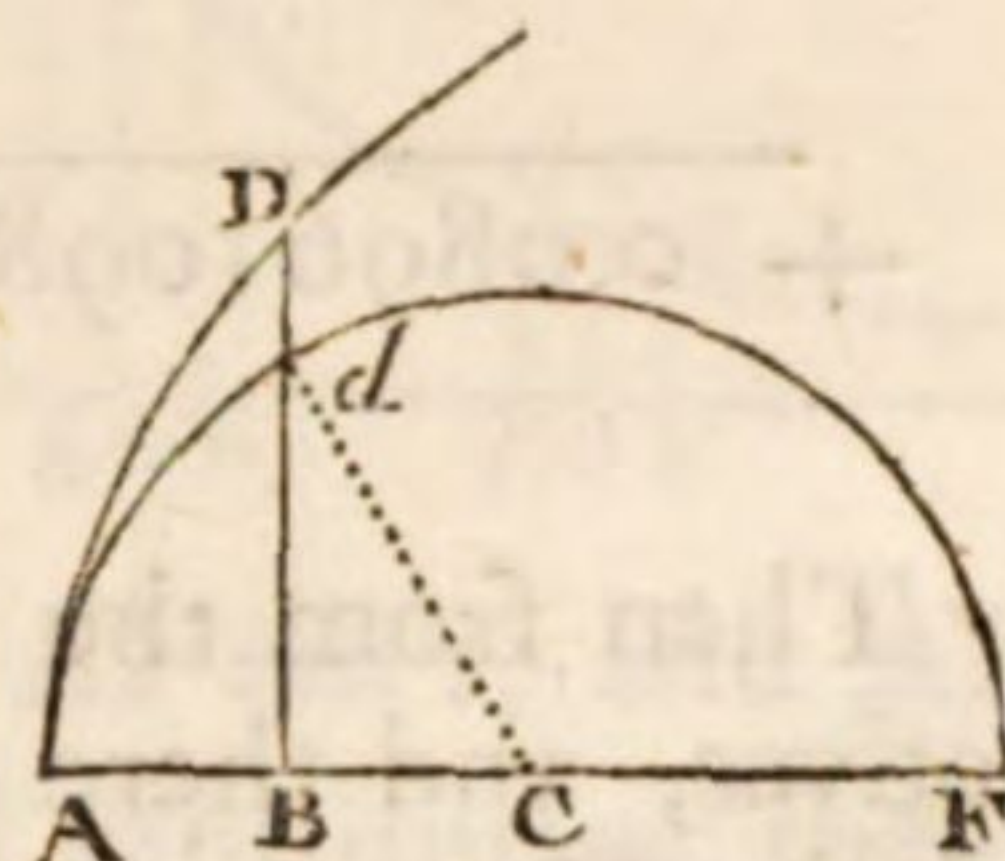
44. Also the Equation $\frac{a^3 - a^2x}{ax + xx} = z$ might have been resolved
 into the two infinite Series $z = \frac{a^3}{x^2} - \frac{a^4}{x^3} + \frac{a^5}{x^4}$, &c. $- a + x$
 $- \frac{xx}{a} + \frac{x^3}{a^2}$, &c. where there is found no Term divided by the first

Power of x . But such kind of Series, where the Powers of x ascend infinitely in the numerators of the one, and in the denominators of the other, are not so proper to derive the value of z from, by Arithmetical computation, when the Species are to be changed into Numbers.

45. Hardly any thing difficult can occur to any one, who is to undertake such a computation in Numbers, after the value of the Area is obtain'd in Species. Yet for the more compleat illustration of the foregoing Doctrine, I shall add an Example or two.

46. Let the Hyperbola AD be proposed, whose Equation is $\sqrt{x+xx}=z$; its Vertex being at A, and each of its Axes is equal to Unity.

From what goes before, its Area ADB $= \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} + \frac{1}{27}x^{\frac{9}{2}} - \frac{5}{704}x^{\frac{11}{2}}$, &c. that is $x^{\frac{1}{2}}$ into $\frac{2}{3}x + \frac{1}{5}x^2 - \frac{1}{28}x^3 + \frac{1}{27}x^4 - \frac{5}{704}x^5$, &c. which Series may be infinitely produced by



multiplying the last term continually by the succeeding terms of this Progression $\frac{1.3}{2.5}x$. $\frac{-1.5}{4.7}x$. $\frac{-3.7}{6.9}x$. $\frac{-5.9}{8.11}x$. $\frac{-7.11}{10.13}x$. &c. That is,

the first term $\frac{2}{3}x^{\frac{3}{2}} \times \frac{1.3}{2.5}x$ makes the second term $\frac{1}{5}x^{\frac{5}{2}}$: Which

multiply'd by $\frac{-1.5}{4.7}x$ makes the third term $-\frac{1}{28}x^{\frac{7}{2}}$: Which mul-

tiple'd by $\frac{-3.7}{6.9}x$ makes $\frac{1}{27}x^{\frac{9}{2}}$ the fourth term ; and so *ad infinitum*. Now let AB be assumed of any length, suppose $\frac{1}{4}$, and writing

this Number for x , and its Root $\frac{1}{2}$ for $x^{\frac{1}{2}}$, and the first term $\frac{2}{3}x^{\frac{3}{2}}$ or $\frac{2}{3} \times \frac{1}{8}$, being reduced to a decimal Fraction, it becomes 0.08333333, &c. This into $\frac{1.3}{2.5 \cdot 4}$ makes 0.00625 the second term.

This into $\frac{-1.5}{4 \cdot 7 \cdot 4}$ makes -0.0002790178 , &c. the third term. And

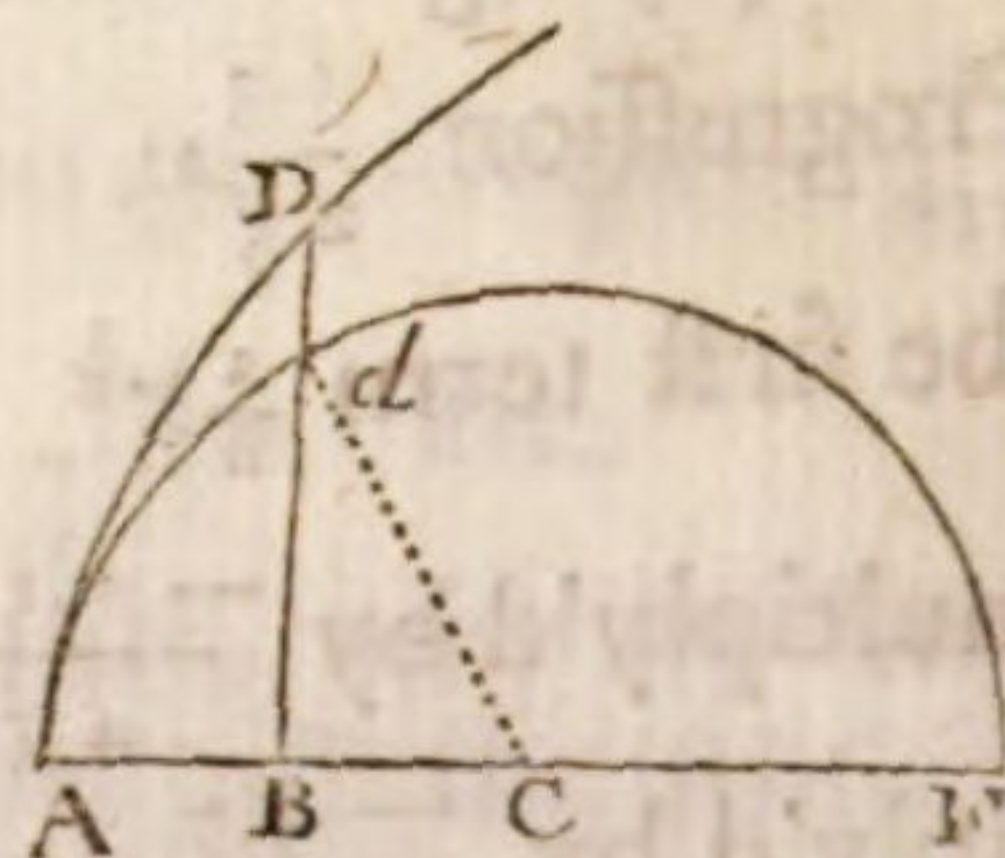
so on for ever. But the terms, which I thus deduce by degrees, I dispose in two Tables ; the affirmative terms in one, and the negative in another, and I add them up as you see here.

+ 0.

+ 0.0833333333333333	— 0.0002790178571429
6250000000000000	34679066051
271267361111	834465027
5135169396	26285354
144628917	961296
4954581	38676
190948	1663
7963	75
352	4
16	
I	
+ 0.0896109885646518	— 0.0002825719389575
	+ 0.0896109885646518
	0.0893284166257043

Then from the sum of the Affirmatives I take the sum of the negatives, and there remains 0.0893284166257043 for the quantity of the Hyperbolic Area ADB; which was to be found.

47. Now let the Circle AdF be proposed, which is expressed by the equation $\sqrt{x - xx} = z$; that is, whose Diameter is unity, and from what goes before its Area AdB will be $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{7}x^{\frac{7}{2}} - \frac{1}{9}x^{\frac{9}{2}}$, &c. In which Series, since the terms do not differ from the terms of the Series, which above express'd the Hyperbolic Area, unless in the Signs + and —; nothing else remains to be done, than to connect the same numeral terms with other signs; that is, by subtracting the connected sums of both the afore-mention'd tables, 0.0898935605036193 from the first term doubled 0.16666666666666, &c. and the remainder 0.0767731061630473 will be the portion AdB of the circular Area, supposing AB to be a fourth part of the diameter. And hence we may observe, that tho' the Areas of the Circle and Hyperbola are not compared in a Geometrical consideration, yet each of them is discover'd by the same Arithmetical computation.



48. The portion of the circle AdB being found, from thence the whole Area may be derived. For the Radius dC being drawn, multiply Bd, or $\frac{1}{4}\sqrt{3}$, into BC, or $\frac{1}{4}$, and half of the product $\frac{1}{3}\sqrt{3}$, or 0.0541265877365275 will be the value of the Triangle CdB; which added to the Area AdB, there will be had the Sector ACd = 0.1308996938995747, the sextuple of which 0.7853981633974482 is the whole Area.

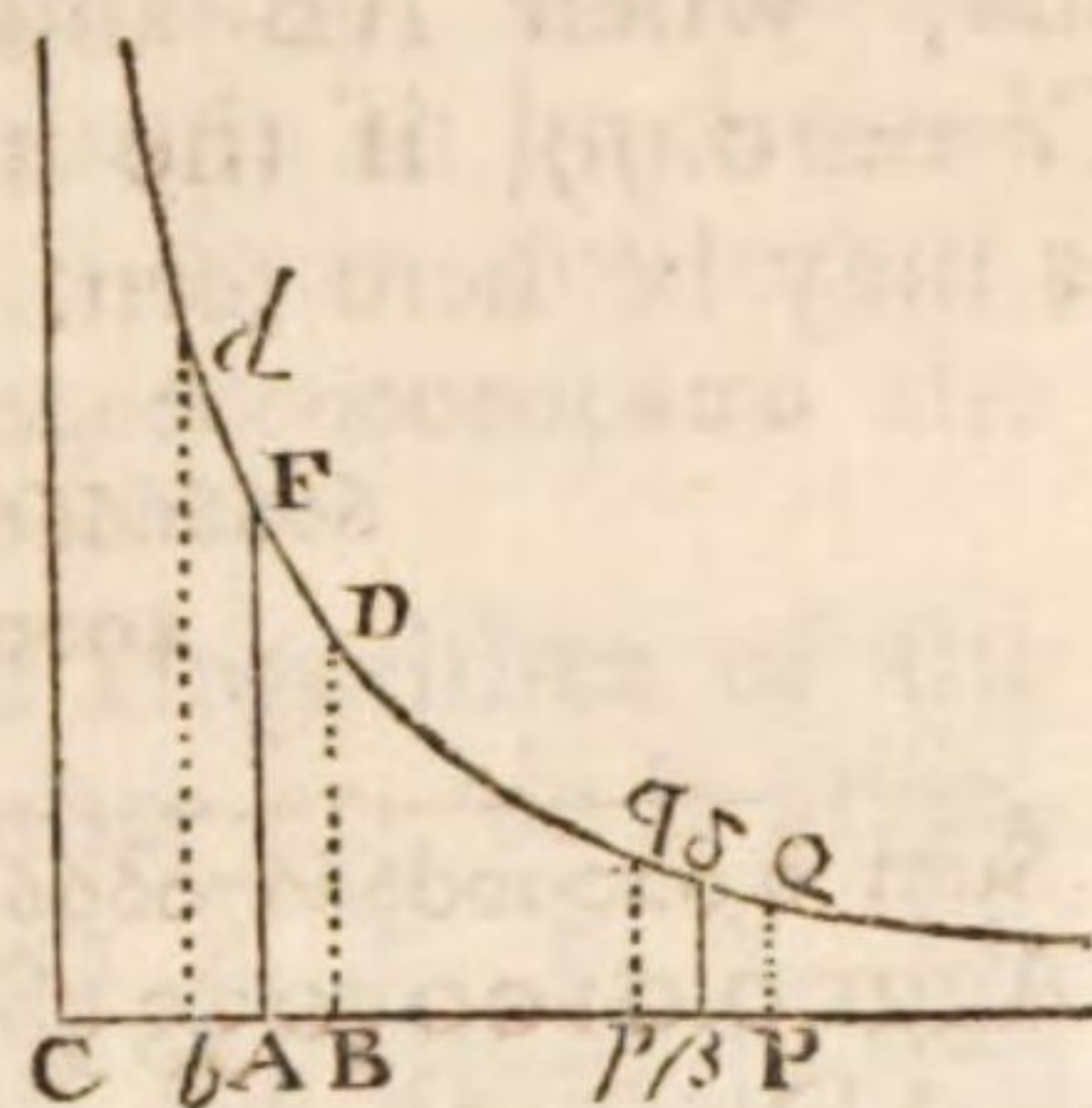
49. And

49. And hence by the way the length of the Circumference will be 3.1415926535897928, by dividing the Area by a fourth part of the Diameter.

50. To these we shall add the calculation of the Area comprehended between the Hyperbola dFD and its Asymptote CA . Let C be the Center of the Hyperbola, and putting $CA = a$, $AF = b$, and $AB = Ab = x$;

'twill be $\frac{ab}{a+x} = BD$, and $\frac{ab}{a-x} = bd$; whence the Area $AFDB = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3}$,

&c. and the Area $AFdb = bx + \frac{bx^2}{2a} + \frac{bx^3}{3a^2} + \frac{bx^4}{4a^3}$, &c. and the sum $bdDB = 2bx + \frac{2bx^3}{3a^2}$ + $\frac{2bx^5}{5a^4} + \frac{2bx^7}{7a^6}$, &c. Now let us suppose $CA = AF = 1$, and Ab or $AB = \frac{1}{10}$, Cb being 0.9, and $CB = 1.1$; and substituting these numbers for a , b , and x , the first term of the Series becomes 0.2, the second 0.0006666666, &c. the third 0.000004; and so on, as you see in this Table.



0.2000000000000000
6666666666666666
400000000000
285714286
2222222
18182
154
1

The sum 0.2006706954621511 = Area $bdDB$.

51. If the parts of this Area Ad and AD be desired separately, subtract the lesser BA from the greater dA , and there will remain $\frac{bx^2}{a} + \frac{bx^4}{2a^3} + \frac{bx^6}{3a^5} + \frac{bx^8}{4a^7}$, &c. Where if 1 be wrote for a and b , and $\frac{1}{10}$ for x , the terms being reduced to decimals will stand thus;

0.0100000000000000
500000000000
3333333333
25000000
200000
1667
14

The sum 0.0100503358535014 = $Ad - AD$.

52. Now if this difference of the Areas be added to, and subtracted from, their sum before found, half the aggregate 0.1053605156578263 will be the greater Area Ad , and half of the remainder 0.0953101798043248 will be the lesser Area AD .

53. By the same tables those Areas AD and Ad will be obtain'd also, when AB and Ab are suppos'd $\frac{1}{100}$, or $CB = 1.01$, and $Cb = 0.99$, if the numbers are but duly transferr'd to lower places, as may be here seen.

$$\begin{array}{r} 0.020000000000000000 \\ 666666666666666666 \\ 40000000 \\ 28 \end{array}$$

$$\begin{array}{r} 0.000100000000000000 \\ 50000000 \\ 3333 \end{array}$$

$$\text{Sum } 0.0001000050003333 \quad Ad - AD.$$

$$\text{Sum } 0.0200006667066695 = bD.$$

$$\frac{1}{2} \text{ Aggr. } 0.0100503358535014 = Ad, \text{ and } \frac{1}{2} \text{ Refid. } 0.0099503308531681 = AD.$$

54. And so putting AB and $Ab = \frac{1}{1000}$, or $CB = 1.001$, and $Cb = 0.999$, there will be obtain'd $Ad = 0.0010005003335835$, and $AD = 0.0009995003330835$.

55. In the same manner (if CA and $AF = 1$) putting AB and $Ab = 0.2$, or 0.02 , or 0.002 , these Areas will arise,

$$\begin{array}{l} Ad = 0.2231435513142097, \text{ and } AD = 0.1823215567939546, \\ \text{or } Ad = 0.0202027073175194, \text{ and } AD = 0.0198026272961797, \\ \text{or } Ad = 0.002002 \quad \text{and } AD = 0.001 \end{array}$$

56. From these Areas thus found it will be easy to derive others, by addition and subtraction alone. For as it is $\frac{1.2}{0.8}$ into $\frac{1.2}{0.9} = 2$, the sum of the Areas 0.6931471805599453 belonging to the Ratio's $\frac{1.2}{0.8}$ and $\frac{1.2}{0.9}$, (that is, insisting upon the parts of the Absciss $1.2 - 0.8$ and $1.2 - 0.9$;) will be the Area $AF\delta\beta$, $C\beta$ being $= 2$, as is known.

Again, since $\frac{1.2}{0.8}$ into $2 = 3$, the sum 1.0986122886681097 of the Area's belonging to $\frac{1.2}{0.8}$ and 2 , will be the Area $AF\delta\beta$, $C\beta$ being 3 .

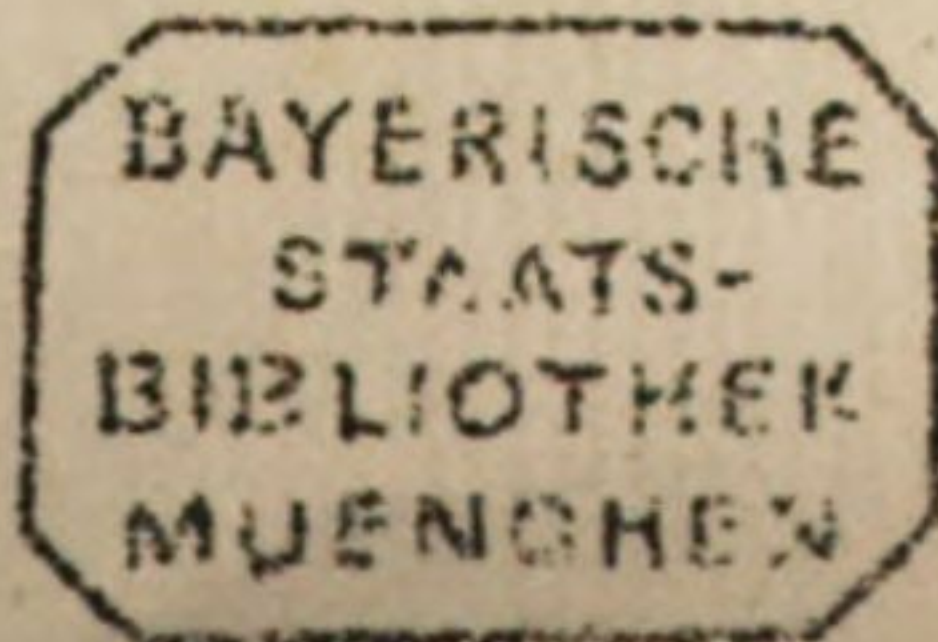
Again, as it is $\frac{2 \times 2}{0.8} = 5$, and $2 \times 5 = 10$, by a due addition of Areas will be obtain'd $1.6093379124341004 = AF\delta\beta$, when $C\beta = 5$; and $2.3025850929940457 = AF\delta\beta$, when $C\beta = 10$. And thus, since $10 \times 10 = 100$, and $10 \times 100 = 1000$, and $\sqrt{5} \times 10 \times 0.98 = 7$, and $10 \times 1.1 = 11$, and $\frac{1000 \times 1.001}{7 \times 11} = 13$, and $\frac{100 \times 0.998}{2} = 499$; it is plain, that the Area $AF\delta\beta$ may be found by the composition of the Areas found before, when $C\beta = 100$; 1000 ;

7; or any other of the above-mention'd numbers, $AB = BF$ being still unity. This I was willing to insinuate, that a method might be derived from hence, very proper for the construction of a Canon of Logarithms, which determines the Hyperbolical Areas, (from which the Logarithms may easily be derived,) corresponding to so many Prime numbers, as it were by two operations only, which are not very troublesome. But whereas that Canon seems to be derivable from this fountain more commodiously than from any other, what if I should point out its construction here, to compleat the whole?

57. First therefore having assumed 0 for the Logarithm of the number 1, and 1 for the Logarithm of the number 10, as is generally done, the Logarithms of the Prime numbers 2, 3, 5, 7, 11, 13, 17, 37, are to be investigated, by dividing the Hyperbolical Areas now found by 2.3025850929940457, which is the Area corresponding to the number 10: Or which is the same thing, by multiplying by its reciprocal 0.4342944819032518. Thus for Instance, if 0.69314718, &c. the Area corresponding to the number 2, were multiply'd by 0.43429, &c. it makes 0.3010299956639812 the Logarithm of the number 2.

58. Then the Logarithms of all the numbers in the Canon, which are made by the multiplication of these, are to be found by the addition of their Logarithms, as is usual. And the void places are to be interpolated afterwards, by the help of this Theorem.

59. Let n be a Number to which a Logarithm is to be adapted, x the difference between that and the two nearest numbers equally distant on each side, whose Logarithms are already found, and let d be half the difference of the Logarithms. Then the required Logarithm of the Number n will be obtain'd by adding $d + \frac{dx}{2n} + \frac{dx^3}{12n^3}$, &c. to the Logarithm of the lesser number. For if the numbers are expounded by Cp , $C\beta$, and CP , the rectangle CBD or $C\beta\delta = 1$, as before, and the Ordinates pq and PQ being raised; if n be wrote for $C\beta$, and x for βp or βP , the Area $pqQP$ or $\frac{2x}{n} + \frac{2x^3}{3n^3} + \frac{2x^5}{5n^5}$, &c. will be to the Area $pq\delta\beta$ or $\frac{x}{n} + \frac{x^2}{2n^2} + \frac{x^3}{3n^3}$, &c. as the difference between the Logarithms of the extream numbers or $2d$, to the difference between the Logarithms of the lesser and of the middle one;



one; which therefore will be $\frac{\frac{dx}{n} + \frac{dx^2}{2n^2} + \frac{dx^3}{3n^3} \&c.}{\frac{x}{n} + \frac{x^3}{3n^3} + \frac{x^5}{5n^5} \&c.}$, that is, when the division is perform'd, $d + \frac{dx}{2n} + \frac{dx^2}{12n^2} \&c.$

60. The two first terms of this Series $d + \frac{dx}{2n}$ I think to be accurate enough for the construction of a Canon of Logarithms, even tho' they were to be produced to fourteen or fifteen figures; provided the number, whose Logarithm is to be found, be not less than 1000. And this can give little trouble in the calculation, because x is generally an unit, or the number 2. Yet it is not necessary to interpolate all the places by the help of this Rule. For the Logarithms of numbers which are produced by the multiplication or division of the number last found, may be obtain'd by the numbers whose Logarithms were had before, by the addition or subtraction of their Logarithms. Moreover by the differences of the Logarithms, and by their second and third differences, if there be occasion, the void places may be more expeditiously supply'd; the foregoing Rule being to be apply'd only, when the continuation of some full places is wanted, in order to obtain those differences.

61. By the same method rules may be found for the intercalation of Logarithms, when of three numbers the Logarithms of the lesser and of the middle number are given, or of the middle number and of the greater; and this although the numbers should not be in Arithmetical progression.

62. Also by pursuing the steps of this method, rules might be easily discover'd, for the construction of the tables of artificial Sines and Tangents, without the assistance of the natural Tables. But of these things only by the bye.

63. Hitherto we have treated of the Quadrature of Curves, which are express'd by Equations consisting of complicate terms; and that by means of their reduction to Equations, which consist of an infinite number of simple terms. But whereas such Curves may sometimes be squared by finite Equations also, or however may be compared with other Curves, whose Areas in a manner may be consider'd as known; of which kind are the Conic Sections: For this reason I thought fit to adjoin the two following catalogues or tables of Theorems, according to my promise, constructed by the help of the 7th and 8th foregoing Propositions.

64. The first of these exhibits the Areas of such Curves as can be squared; and the second contains such Curves, whose Areas may be compared with the Areas of the Conic Sections. In each of these, the letters d, e, f, g , and h , denote any given quantities, x and z the Abscisses of Curves, v and y parallel Ordinates, and s and t Areas, as before. The letters n and θ , annex'd to the quantity z , denote the number of the dimensions of the same z , whether it be integer or fractional, affirmative or negative. As if $n=3$, then $z^n=z^3$, $z^{2n}=z^6$, $z^{-n}=z^{-3}$ or $\frac{1}{z^3}$, $z^{n+1}=z^4$, and $z^{n-1}=z^2$.

65. Moreover in the values of the Areas, for the sake of brevity, is written R instead of this Radical $\sqrt{e+fz^n}$, or $\sqrt{e+fz^n+gz^{2n}}$, and p instead of $\sqrt{b+iz^n}$, by which the value of the Ordinate y is affected.

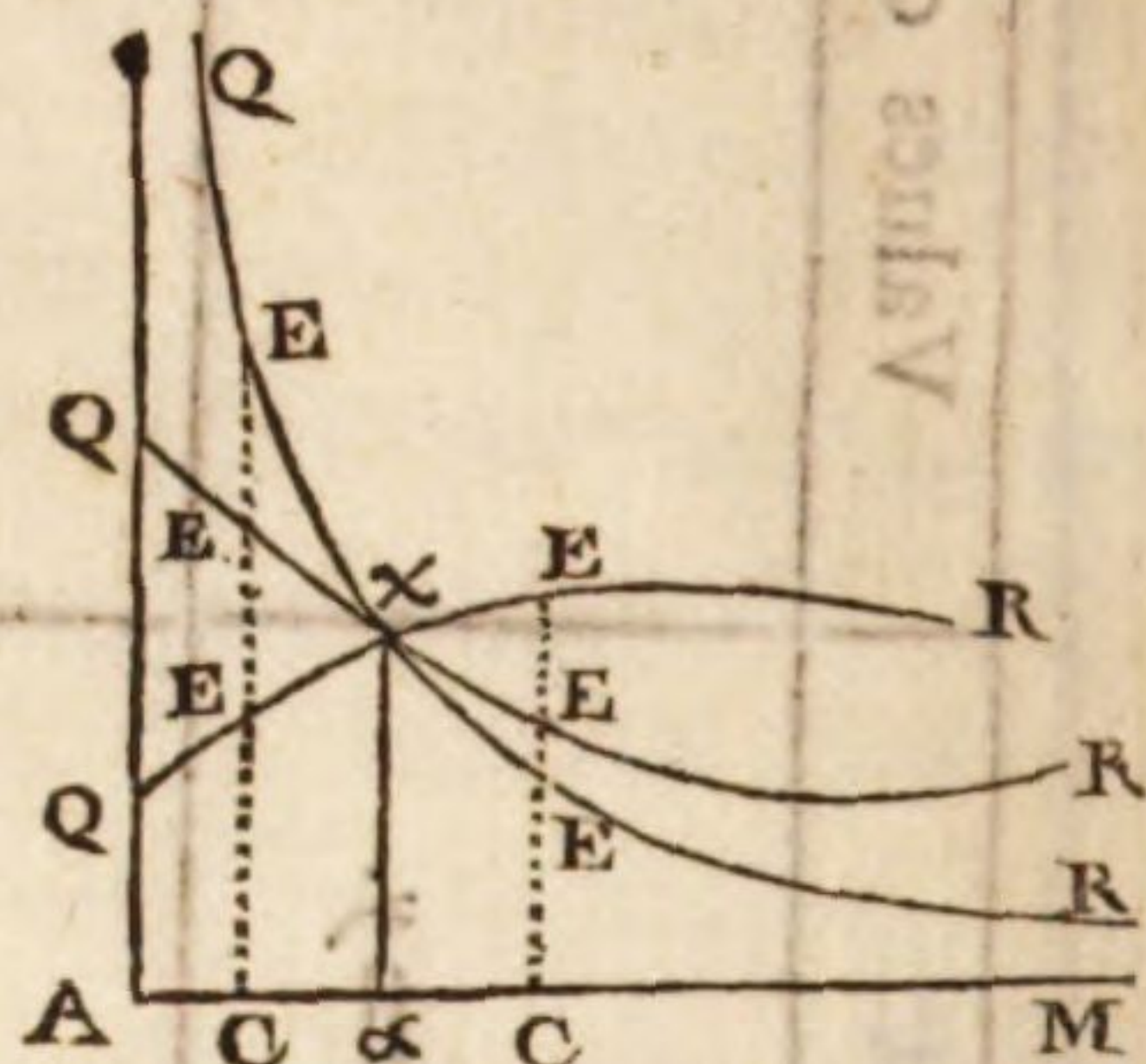
$A = \frac{Rb}{n} + \frac{fz^{n+1}}{n+1} + \frac{gz^{2n+1}}{2n+1}$	$y = \frac{R}{p} + \frac{fz^n}{p^2} + \frac{gz^{2n}}{p^3}$	I
$A = \frac{Rb}{n} + \frac{fz^{n+1}}{n+1} + \frac{gz^{2n+1}}{2n+1}$	$y = \frac{R}{p} + \frac{fz^n}{p^2} + \frac{gz^{2n}}{p^3}$	II
$A = \frac{Rb}{n} + \frac{fz^{n+1}}{n+1} + \frac{gz^{2n+1}}{2n+1}$	$y = \frac{R}{p} + \frac{fz^n}{p^2} + \frac{gz^{2n}}{p^3}$	III
$A = \frac{Rb}{n} + \frac{fz^{n+1}}{n+1} + \frac{gz^{2n+1}}{2n+1}$	$y = \frac{R}{p} + \frac{fz^n}{p^2} + \frac{gz^{2n}}{p^3}$	IV
$A = \frac{Rb}{n} + \frac{fz^{n+1}}{n+1} + \frac{gz^{2n+1}}{2n+1}$	$y = \frac{R}{p} + \frac{fz^n}{p^2} + \frac{gz^{2n}}{p^3}$	V
$A = \frac{Rb}{n} + \frac{fz^{n+1}}{n+1} + \frac{gz^{2n+1}}{2n+1}$	$y = \frac{R}{p} + \frac{fz^n}{p^2} + \frac{gz^{2n}}{p^3}$	VI
$A = \frac{Rb}{n} + \frac{fz^{n+1}}{n+1} + \frac{gz^{2n+1}}{2n+1}$	$y = \frac{R}{p} + \frac{fz^n}{p^2} + \frac{gz^{2n}}{p^3}$	VII

66. The first Table, of some Curvilinear Areas related to Rectilinear Figures, constructed by Prob. 7.

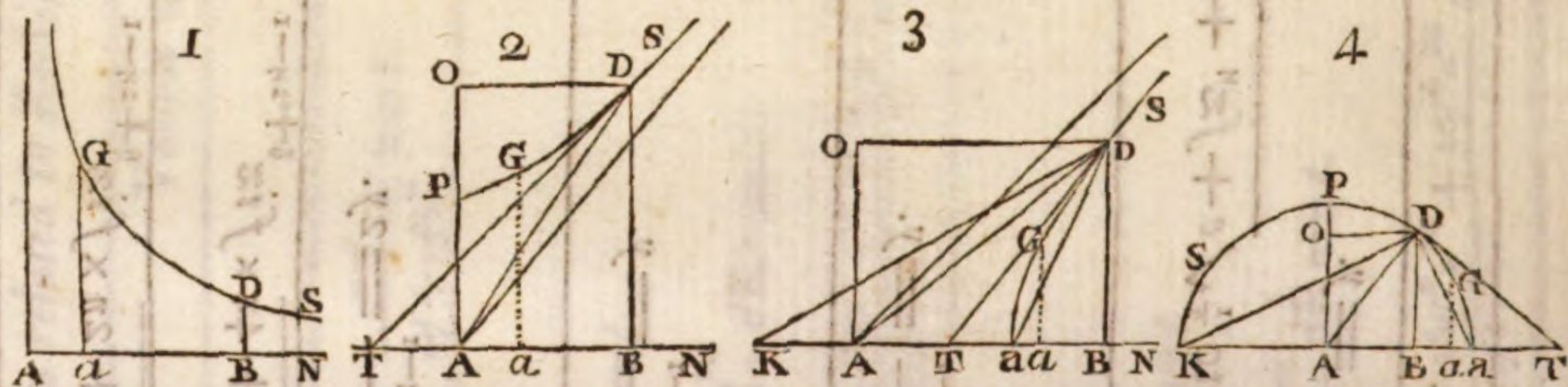
Order of Curves.		Values of their Areas.	
I.	$dx^{n-1} = y$	$\frac{d}{n} x^n = t.$	
II.	$\frac{dx^{n-1}}{e + 2efx^n + ffx^{2n}} = y$	$\frac{dx^n}{ne^2 + nefx^n} = t, \text{ or } \frac{-d}{nef + nffx^n} = t.$	
III.	1 $dx^{n-1} \sqrt{e + f x^n} = y$	$\frac{2d}{3nf} R^3 = t.$	
	2 $dx^{2n-1} \sqrt{e + f x^n} = y$	$\frac{-4e + 6fx^n}{15nff} dR^3 = t.$	
	3 $dx^{3n-1} \sqrt{e + f x^n} = y$	$\frac{16ee - 24efx^n + 30ffx^{2n}}{105nf^3} dR^3 = t.$	
	4 $dx^{4n-1} \sqrt{e + f x^n} = y$	$\frac{-96e^3 + 144e^2fx^n - 180ef^2x^{2n} + 210f^3x^{3n}}{945nf^4} dR^3 = t.$	
IV.	1 $\frac{dx^{n-1}}{\sqrt{e + f x^n}} = y$	$\frac{2d}{nf} R = t.$	
	2 $\frac{dx^{2n-1}}{\sqrt{e + f x^n}} = y$	$\frac{-4e + 2fx^n}{3nff} dR = t.$	
	3 $\frac{dx^{3n-1}}{\sqrt{e + f x^n}} = y$	$\frac{16e^2 - 8efx^n + 6ffx^{2n}}{15nf^3} dR = t.$	
	4 $\frac{dx^{4n-1}}{\sqrt{e + f x^n}} = y$	$\frac{-96e^3 + 48e^2fx^n - 36ef^2x^{2n} + 30f^3x^{3n}}{105nf^4} dR = t.$	

	Order of Curves.	Values of Areas.
V.	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ez^{\theta-1} + 2\theta fz^{\theta+1-1} \\ + 3nf^{\theta+1-1} \end{array}$	$\int \frac{1}{2} \sqrt{e + fz^{\theta}} = y.$
	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ez^{\theta-1} + 2\theta fz^{\theta+1-1} \\ + 2\theta gz^{\theta+2-1} + 6ng^{\theta+2-1} \end{array}$	$\int \frac{1}{2} \sqrt{e + fz^{\theta} + gz^{2\theta}} = y.$
VI.	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ez^{\theta-1} + 2\theta + n \times fz^{\theta+1-1} \\ + 2\theta + 2n \times gz^{\theta+2-1} \end{array}$	$\int y = y.$
	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ez^{\theta-1} + 2\theta + n \times fz^{\theta+1-1} \\ + 2\theta + 2n \times gz^{\theta+2-1} \end{array}$	$\int y = y.$
VII.	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ez^{\theta-1} + 2\theta - n \times fz^{\theta+1-1} \\ + 2\theta - 2n \times gz^{\theta+2-1} \end{array}$	$\int y = y.$
	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ez^{\theta-1} + 2\theta - n \times fz^{\theta+1-1} \\ + 2\theta - 2n \times gz^{\theta+2-1} \end{array}$	$\int y = y.$
VIII.	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ez^{\theta-1} + 2\theta - 2n \times fz^{\theta+1-1} \\ + 2\theta - 4n \times gz^{\theta+2-1} \end{array}$	$\int y = 2y.$
	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ez^{\theta-1} + 2\theta - 2n \times fz^{\theta+1-1} \\ + 2\theta - 4n \times gz^{\theta+2-1} \end{array}$	$\int y = 2y.$
IX.	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ebz^{\theta-1} + 2\theta + 3n \times fbz^{\theta+1-1} \\ + 2\theta + n \times ei^{\theta+2-1} \end{array}$	$\int \frac{\sqrt{e + fz^{\theta}}}{2\sqrt{b + iz^{\theta}}} = y.$
	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ebz^{\theta-1} + 2\theta + 3n \times fbz^{\theta+1-1} \\ + 2\theta + n \times ei^{\theta+2-1} \end{array}$	$\int \frac{\sqrt{e + fz^{\theta}}}{b + iz^{\theta}} \text{ into } 2\sqrt{b + iz^{\theta}} = y.$
X.	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ebz^{\theta-1} + 2\theta + 3n \times fbz^{\theta+1-1} \\ + 2\theta + n \times ei^{\theta+2-1} \end{array}$	$\int \frac{\sqrt{e + fz^{\theta}}}{b + iz^{\theta}} \text{ into } 2\sqrt{b + iz^{\theta}} = y.$
	$\left. \begin{array}{l} 1 \\ 2 \end{array} \right\} \begin{array}{l} 2\theta ebz^{\theta-1} + 2\theta + 3n \times fbz^{\theta+1-1} \\ + 2\theta + n \times ei^{\theta+2-1} \end{array}$	$\int \frac{\sqrt{e + fz^{\theta}}}{b + iz^{\theta}} \text{ into } 2\sqrt{b + iz^{\theta}} = y.$

67. Other things of the same kind might have been added; but I shall now pass on to another sort of Curves, which may be compared with the Conic Sections. And in this Table or Catalogue you have the proposed Curve represented by the Line $QE\chi R$, the beginning of whose Absciss is A, the Absciss AC, the Ordinate CE, the beginning of the Area $\alpha\chi$, and the Area described $\alpha\chi EC$. But the beginning of this Area, or the initial term, (which commonly either commences at the beginning of the Absciss A, or recedes to an infinite distance,) is found by seeking the length of the Absciss $A\alpha$, when the value of the Area is nothing, and by erecting the perpendicular $\alpha\chi$.



68. After the same manner you have the Conic Section represented by the Line PDG, whose Center is A, Vertex a, rectangular



Semidiameters Aa and AP, the beginning of the Absciss A, or a, or α , the Absciss AB, or aB, or αB , the Ordinate BD, the Tangent DT meeting AB in T, the Subtense aD, and the Rectangle inscribed or adscribed ABDO.

69. Therefore retaining the letters before defined, it will be $AC = z$, $CE = y$, $\alpha\chi EC = t$, AB or aB = x , BD = v , and ABDP or aGDB = s . And besides, when two Conic Sections are required, for the determination of any Area, the Area of the latter shall be call'd σ , the Absciss ξ , and the Ordinate τ . Put p for $\sqrt{ff - 4eg}$.

70. The second Table, of some Curvilinear Areas, related to the Conic Sections, constructed by the help of Prob. 8.

Forms of Curves.		Conic Sections.		Values of the Areas.	
		Abscis.	Ordinate.		
I.	$\frac{dz^{\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fz} = v$	$\frac{1}{\eta} s = t = \frac{aGDB}{\eta}$	Fig. 1.
	$\frac{dz^{2\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fz} = v$	$\frac{d}{\eta f} z^{\eta} - \frac{e}{\eta f} s = t.$	
	$\frac{dz^{3\eta-1}}{e+fz^{\eta}} = y$	$z^{\eta} = x$	$\frac{d}{e+fz} = v$	$\frac{d}{2\eta f} z^{2\eta} - \frac{de}{\eta f^2} z^{\eta} + \frac{e^2}{\eta f^2} s = t.$	
II.	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}} = y$	$\sqrt{\frac{d}{e+fz^{\eta}}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f^2} x^2} = v$	$\frac{2xv}{\eta} - \frac{4s}{\eta} = t = \frac{4}{\eta} ADGa.$	Fig. 3, 4.
	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}} = y$	$\sqrt{\frac{d}{e+fz^{\eta}}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f^2} x^2} = v$	$\frac{2d}{\eta f} z^{\frac{1}{2}\eta} + \frac{4es - 2exv}{\eta f} = t.$	
	$\frac{dz^{\frac{5}{2}\eta-1}}{e+fz^{\eta}} = y$	$\sqrt{\frac{d}{e+fz^{\eta}}} = x$	$\sqrt{\frac{d}{f} - \frac{e}{f^2} x^2} = v$	$\frac{2d}{3\eta f} z^{\frac{3}{2}\eta} - \frac{2de}{\eta f^2} z^{\frac{1}{2}\eta} + \frac{2e^2 xv - 4e^2 s}{\eta f^2} = t.$	
III.	$\frac{1}{z^{\eta}} \sqrt{e+fz^{\eta}} = y,$ or thus	$\frac{1}{z^{\eta}} = x^2$	$\sqrt{f+ex^2} = v$	$\frac{4de}{\eta f} \times \frac{v^3}{2ex} - s = t = \frac{4de}{\eta f}$ into aGDT, or into APDB ÷ TDB. [Fig. 2, 3, 4.]	
		$\frac{1}{z^{\eta}} = x$	$\sqrt{fx+ex^2} = v$	$\frac{8de^2}{\eta f^2} \times s - \frac{1}{2} xv - \frac{fv}{4e} + \frac{f^2 v}{4e^2 x} = t = \frac{8de^2}{\eta f^2}$ into aGDA + $\frac{f^2 v}{4e^2 x}$ [Fig. 3, 4.]	
	$\frac{d}{z^{\eta+1}} \sqrt{e+fz^{\eta}} = y$	$\frac{1}{z^{\eta}} = x^2$	$\sqrt{f+ex^2} = v$	$-\frac{2d}{\eta} s = t = \frac{2d}{\eta}$ APDB or $\frac{2d}{\eta}$ aGDB.	Fig. 2, 3, 4.
	or thus	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx+ex^2} = v$	$\frac{4de}{\eta f} \times s - \frac{1}{2} xv - \frac{fv}{2e} = t = \frac{4de}{\eta f}$ x aGDK.	Fig. 3, 4.
	$\frac{d}{z^{\eta+1}} \sqrt{e+fz^{\eta}} = y$	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx+ex^2} = v$	$-\frac{d}{\eta} s = t = \frac{d}{\eta} x - \frac{aGDB}{\eta}$ or BDPK.	Fig. 4.
	$\frac{d}{z^{3\eta+1}} \sqrt{e+fz^{\eta}} = y$	$\frac{1}{z^{\eta}} = x$	$\sqrt{fx+ex^2} = v$	$\frac{3dfs - 2dv^3}{6\eta e} = t.$	

Forms of Curves.	Conic Sections. Abscifs.	Ordinate.	Values of the Areas.
IV.	1	$\frac{d}{x^2} = x^2$	$\frac{4d}{nf} \times \frac{x^2}{2} = t = \frac{4d}{nf}$ into PAD or into aGDA. Fig. 2, 3, 4
	2	$\frac{d}{x^2} = x^2$ or thus	$\frac{8de}{nf^2} \times \frac{1}{2} x^2 = t = \frac{8de}{nf^2}$ into aGDA. Fig. 3, 4
	3	$\frac{d}{x^2} = x^2$ or thus	$\frac{2d}{ne} \times \frac{1}{2} x^2 = t = \frac{2d}{ne}$ into POD or into AODGa. Fig. 2, 3, 4
	4	$\frac{d}{x^2} = x^2$	$\frac{4d}{nf} \times \frac{1}{2} x^2 = t = \frac{4d}{nf}$ into aDGA. Fig. 3, 4
V.	1	$\frac{d}{x^2} = x^2$	$\frac{d}{ne} \times 3s = 2xv = t = \frac{d}{ne}$ into 3aDGA = ΔaDB . Fig. 3, 4
	2	$\frac{d}{x^2} = x^2$	$\frac{10dfxv}{6ne} = 15dfg - 2dex^2v = t$
	3	$\frac{d}{x^2} = x^2$	$\frac{xv - 2s}{n} = t$
	4	$\frac{d}{x^2} = x^2$	$\frac{fs - xv}{n} = t$
VI.	1	$\frac{d}{x^2} = x^2$	$\frac{d\sigma + 2fs - fxv}{2ng} = t$
	2	$\frac{d}{x^2} = x^2$	$\frac{2xv - 4s - 2\sigma + 4\sigma}{np} = t$
	3	$\frac{d}{x^2} = x^2$	$\frac{4s - 2xv - 4\sigma + 2\sigma}{np} = t$
	4	$\frac{d}{x^2} = x^2$	$\frac{4s - 2xv - 4\sigma + 2\sigma}{np} = t$

Forms of Curves.		Conic Sections.		Values of the Areas.	
		Absciss.	Ordinate.		
VII.	1	$\frac{d}{z} \sqrt{e + fz^{\eta} + gz^{2\eta}} = y$	$\left\{ \begin{array}{l} \sqrt{e + fx + gx^2} = v \\ \sqrt{g + f\xi + e\xi^2} = \gamma \end{array} \right\}$	$\frac{4de^2\xi\gamma + 2def\gamma - 2dfgxv - 2ffd}{4neg - nff} + \frac{4degv}{2ffd} - 8de^2\sigma + 4dfg^2 = t.$	Fig. 2, 3, 4.
	2	$dz^{\eta-1} \sqrt{e + fz^{\eta} + gz^{2\eta}} = y$	$\sqrt{e + fx + gx^2} = v$	$\frac{d}{n} s = t = \frac{d}{n} \times \alpha \text{GDB.}$	
	3	$dz^{2\eta-1} \sqrt{e + fz^{\eta} + gz^{2\eta}} = y$	$\sqrt{e + fx + gx^2} = v$	$\frac{d}{3ng} v^3 - \frac{df}{2ng} s = t.$	
	4	$dz^{3\eta-1} \sqrt{e + fz^{\eta} + gz^{2\eta}} = y$	$\sqrt{e + fx + gx^2} = v$	$\frac{6d\eta x - 5df}{24ng^2} v^3 + \frac{5df^2 - 4deg}{16ng^2} s = t.$	
VIII.	1	$\frac{dz^{\eta-1}}{\sqrt{e + fz^{\eta} + gz^{2\eta}}} = y$	$\sqrt{e + fx + gx^2} = v$	$\frac{8d\eta s - 4degxv - 2dfv}{4neg - nff^2} = t = \frac{8dg}{4neg - nff^2} \times \alpha \text{GDB} = \Delta \text{DBA.}$	[Fig. 2, 4.]
	2	$\frac{dz^{2\eta-1}}{\sqrt{e + fz^{\eta} + gz^{2\eta}}} = y$	$\sqrt{e + fx + gx^2} = v$	$\frac{-4dfs + 2dfxv + 4dev}{4neg - nff^2} = t.$	
	3	$\frac{dz^{3\eta-1}}{\sqrt{e + fz^{\eta} + gz^{2\eta}}} = y$	$\sqrt{e + fx + gx^2} = v$	$\frac{3dff - 2dff}{-4deg + 4deg} xv - 2defv = t.$	
	4	$\frac{dz^{4\eta-1}}{\sqrt{e + fz^{\eta} + gz^{2\eta}}} = y$	$\sqrt{e + fx + gx^2} = v$	$\frac{36defg + 8degx^2v + 10dff}{-15df^3 - 2df^2g - 28defg} xv + \frac{10dff}{-28defg} xv + \frac{10defv}{-16de^2g} v = t.$	
IX.	1	$\frac{dz^{\eta-1}}{g + bz^{\eta}} \sqrt{e + fz^{\eta}} = y$	$\sqrt{\frac{df}{b} + \frac{eb - fg}{b} x^2} = v$	$\frac{4fg}{-4eb} s + \frac{2fg}{nfb} xv + 2df \frac{v}{x} = t.$	
	2	$\frac{dz^{2\eta-1}}{g + bz^{\eta}} \sqrt{e + fz^{\eta}} = y$	$\sqrt{\frac{df}{b} + \frac{eb - fg}{b} x^2} = v$	$\frac{4egb}{-4fgg} s + \frac{2egb}{2fgg} xv + \frac{2db}{nfb^2} \frac{v^3}{x^3} - 2dfg \frac{v}{x} = t.$	

Forms of Curves.	Abfcifs.	Conic Sections. Ordinate.	Values of the Areas.
X.	$\sqrt{\frac{dz^{2n-1}}{g+bz^n}} = y$	$\sqrt{\frac{df}{b} + \frac{eb-fg}{b}x^2} = v$	$\frac{2xv-4s}{vf} = t = \frac{4}{vf} \text{ ADGa.}$
	$\sqrt{\frac{dz^{2n-1}}{g+bz^n}} = y$	$\sqrt{\frac{df}{b} + \frac{eb-fg}{b}x^2} = v$	$\frac{1gs-2gxv+2d}{vf} = t.$
	$\sqrt{\frac{dz^{2n-1}}{g+bz^n}} = y$	$\sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2} = v$	$\frac{2dxv^3z^{-n} - 4dfs - 4des}{nfg-neb} = t.$
XI.	$\sqrt{\frac{dz^{2n-1}}{g+bz^n}} = y$	$\sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2} = v$	$\frac{2d}{nb} s = t.$
	$\sqrt{\frac{dz^{2n-1}}{g+bz^n}} = y$	$\sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2} = v$	$\frac{dbxv^3 - 3dfgs}{2nfb^2} = t.$
	$\sqrt{\frac{dz^{2n-1}}{g+bz^n}} = y$	$\sqrt{\frac{eb-fg}{b} + \frac{f}{b}x^2} = v$	$\frac{dbxv^3 - 3dfgs}{2nfb^2} = t.$

Area of the Area.

71. Before I go on to illustrate by Examples the Theorems that are deliver'd in these classes of Curves, I think it proper to observe,

72. I. That whereas in the Equations representing Curves, I have all along supposed all the signs of the quantities d , e , f , g , b , and i to be affirmative; whenever it shall happen that they are negative, they must be changed in the subsequent values of the Absciss and Ordinate of the Conic Section, and also of the Area required.

73. II. Also the signs of the numeral Symbols n and θ , when they are negative, must be changed in the values of the Areas. Moreover their Signs being changed, the Theorems themselves may acquire a new form. Thus in the 4th Form of Table 2, the Sign of n being changed, the 3d Theorem becomes $\frac{d}{z^{-2n+1}\sqrt{e+fz^{-n}}} = y, \frac{1}{z^{-n}}$

$= x$, &c. that is, $\frac{dx^{3n-1}}{\sqrt{ez^{2n}+fz^n}} = y, z^n = x, \sqrt{fx+ex^2} = v, \frac{d}{n^e}$
into $2xv - 3s = t$. And the same is to be observed in others.

74. III. The series of each order, excepting the 2d of the 1st Table, may be continued each way *ad infinitum*. For in the Series of the 3d and 4th Order of Table 1, the numeral co-efficients of the initial terms, (2, -4, 16, -96, 768, &c.) are form'd by multiplying the numbers -2, -4, -6, -8, -10, &c. continually into each other; and the co-efficients of the subsequent terms are derived from the initials in the 3d Order, by multiplying gradually by $-\frac{3}{2}, -\frac{5}{4}, -\frac{7}{6}, -\frac{9}{8}, -\frac{11}{10},$ &c. or in the 4th Order by multiplying by $-\frac{1}{2}, -\frac{3}{4}, -\frac{5}{6}, -\frac{7}{8}, -\frac{9}{10},$ &c. But the co-efficients of the denominators 1, 3, 15, 105, &c. arise by multiplying the numbers 1, 3, 5, 7, 9, &c. gradually into each other.

75. But in the 2d Table, the Series of the 1st, 2^d, 3^d, 4th, 9th, and 10th Orders are produced *in infinitum* by division alone. Thus having

$\frac{dz^{4n-1}}{e+fz^n} = y$, in the 1st Order, if you perform the division to a convenient period, there will arise $\frac{d}{f} z^{3n-1} - \frac{de}{ff} z^{2n-1} + \frac{de^2}{f^3} z^{n-1} - \frac{de^3}{f^3} z^{n-1} = y$. The first three terms belong to the 1st Order of

Table 1, and the fourth term belongs to the 1st Species of this Order.

Whence it appears, that the Area is $\frac{d}{3nf} z^{3n} - \frac{de}{2nff} z^{2n} + \frac{de^2}{nf^3} z^n - \frac{e^3}{nf^3} s$; putting s for the Area of the Conic Section, whose Absciss

is $x = z^n$, and Ordinate $v = \frac{d}{e+fx}$.

76. But the Series of the 5th and 6th Orders may be infinitely continued, by the help of the two Theorems in the 5th Order of Table 1. by a due addition or subtraction: As also the 7th and 8th Series, by means of the Theorems in the 6th Order of Table 1. and the Series of the 11th, by the Theorem in the 10th Order of Table 1.

For instance, if the Series of the 3d Order of Table 2. be to be farther continued, suppose $\theta = -4n$, and the 1st Theorem of the 5th Order of Table 1. will become $-8ne z^{-4n-1} - 5nf z^{-3n-1}$ into $\frac{1}{2}\sqrt{e+fz^n} = y$. $\frac{R^3}{z^{4n}} = t$. But according to the 4th Theorem of

this Series to be produced, writing $-\frac{5nf}{2}$ for d , it is $-\frac{5n}{2} f z^{-3n-1}$
 $\sqrt{e+fz^n} = y$, $\frac{1}{z^n} = x$, $\sqrt{fx+exx} = v$, and $\frac{10fv^3-15f^2s}{12e} = t$.

So that subtracting the former values of y and t , there will remain

$4ne z^{-4n-1} \sqrt{e+fz^n} = y$, $\frac{10fv^3-15f^2s}{12e} - \frac{R^3}{z^{4n}} = t$. These being mul-

tiplied by $\frac{d}{4ne}$; and, (if you please) for $\frac{R^3}{z^{4n}}$ writing xv^3 , there will arise

a 5th Theorem of the Series to be produced, $\frac{d}{z^{4n+1}} \sqrt{e+fz^n} = y$, $\frac{1}{z^n} = x$,
 $\sqrt{fx+exx} = v$, and $\frac{10dfv^3-15df^2s}{48ne^2} - \frac{dxv^3}{4ne} = t$.

77. IV. Some of these Orders may also be otherwise derived from others. As in the 2d Table, the 5th, 6th, 7th, and 11th, from the 8th; and the 9th from the 10th: So that I might have omitted them, but that they may be of some use, tho' not altogether necessary. Yet I have omitted some Orders, which I might have derived from the 1st, and 2d, as also from the 9th and 10th, because they were affected by Denominators that were more complicate, and therefore can hardly be of any use.

78. V. If the defining Equation of any Curve is compounded of several Equations of different Orders, or of different Species of the same Order, its Area must be compounded of the corresponding Areas; taking care however, that they may be rightly connected with their proper Signs. For we must not always add or subtract at the same time Ordinates to or from Ordinates, or corresponding Areas to or from corresponding Areas; but sometimes the sum of these, and the difference of those, is to be taken for a new Ordinate, or to constitute a corresponding Area. And this must be done, when the constituent Areas are posited on the contrary side of the Ordinate. But that the cautious Geometrician may the more readily avoid this

inconveniency, I have prefix'd their proper Signs to the several Values of the Areas, tho' sometimes negative, as is done in the 5th and 7th Order of Table 2.

79. VI. It is farther to be observed, about the Signs of the Areas, that $+s$ denotes, either that the Area of the Conic Section, adjoining to the Absciss, is to be added to the other quantities in the value of t ; (see the 1st Example following;) or that the Area on the other side of the Ordinate is to be subtracted. And on the contrary, $-s$ denotes ambiguously, either that the Area adjacent to the Absciss is to be subtracted, or that the Area on the other side of the Ordinate is to be added, as it may seem convenient. Also the Value of t , if it comes out affirmative, denotes the Area of the Curve proposed adjoining to its Absciss: And contrariwise, if it be negative, it represents the Area on the other side of the Ordinate.

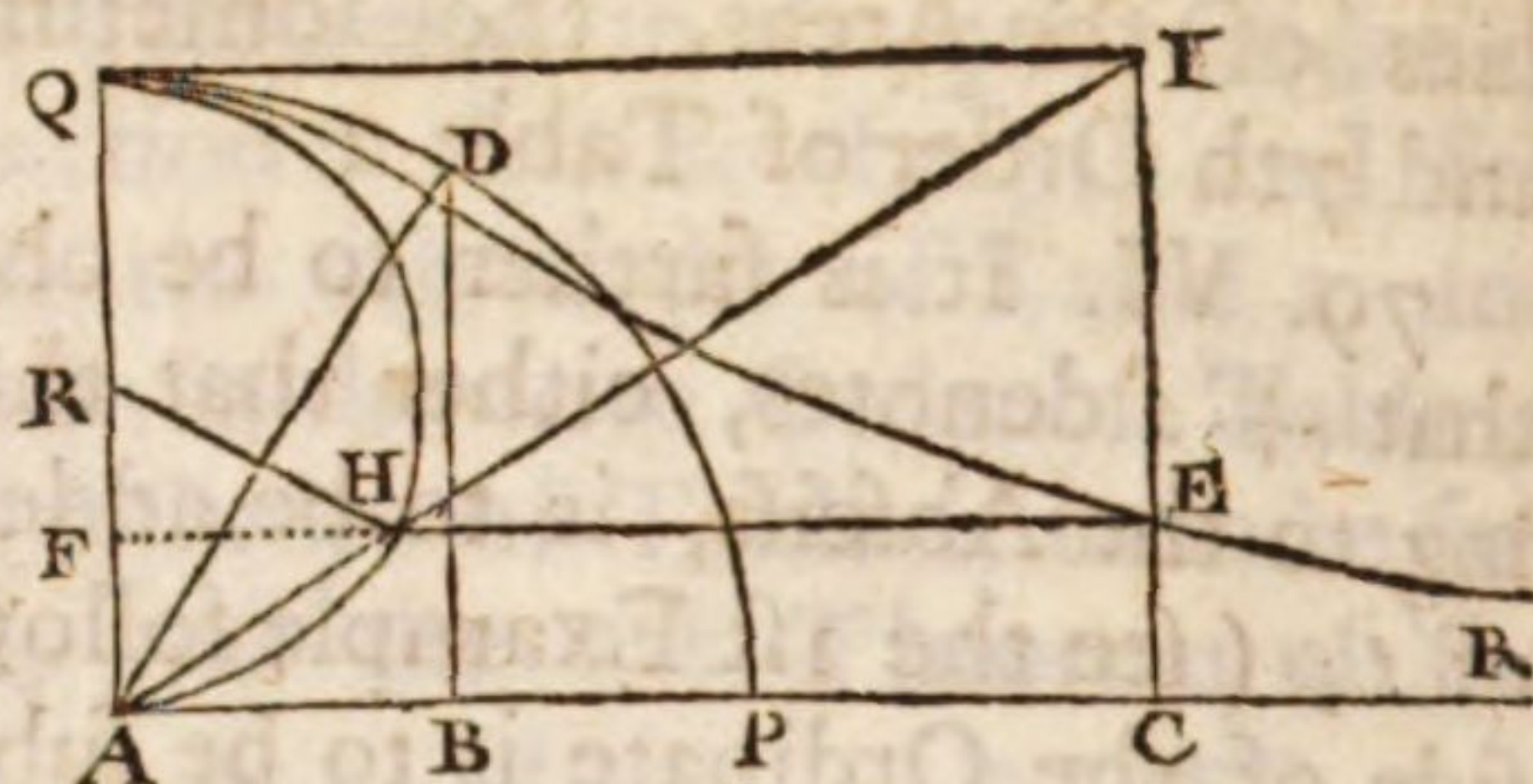
80. VII. But that this Area may be more certainly defined, we must enquire after its Limits. And as to its Limit at the Absciss, at the Ordinate, and at the Perimeter of the Curve, there can be no uncertainty: But its initial Limit, or the beginning from whence its description commences, may obtain various positions. In the following Examples it is either at the beginning of the Absciss, or at an infinite distance, or in the concourse of the Curve with its Absciss. But it may be placed elsewhere. And wherever it is, it may be found, by seeking that length of the Absciss, at which the value of t becomes nothing, and there erecting an Ordinate. For the Ordinate so raised will be the Limit required.

81. VIII. If any part of the Area is posited below the Absciss, t will denote the difference of that, and of the part above the Absciss.

82. IX. Whenever the dimensions of the terms in the values of x , v , and t , shall ascend too high, or descend too low, they may be reduced to a just degree, by dividing or multiplying so often by any given quantity, which may be suppos'd to perform the office of Unity, as often as those dimensions shall be either too high or too low.

83. X. Besides the foregoing Catalogues, or Tables, we might also construct Tables of Curves related to other Curves, which may be the most simple in their kind; as to $\sqrt{a + fx^3} = v$, or to $x\sqrt{e + fx^3} = v$, or to $\sqrt{e + fx^4} = v$, &c. So that we might at all times derive the Area of any proposed Curve from the simplest original, and know to what Curves it stands related. But now let us illustrate by Examples, what has been already delivered.

84. EXAMPLE I. Let QER be a Conchoidal of such a kind, that the Semicircle QHA being described, and AC being erected perpendicular to the Diameter AQ; if the Parallelogram QACI be compleated, the Diagonal AI be drawn, meeting the Semicircle in H, and from H the perpendicular HE be let fall to IC; then the Point E will describe a Curve, whose Area ACEQ is sought.



85. Therefore make $AQ = a$, $AC = z$, $CE = y$; and because of the continual Proportionals AI, AQ, AH, EC, 'twill be EC or $y = \frac{a^3}{a^2 + z^2}$.

86. Now that this may acquire the Form of the Equations in the Tables, make $n = 2$, and for z^2 in the denominator write z^n , and $a^3 z^{\frac{1}{2}n-1}$ for a^3 or $a^3 z^{1-1}$ in the numerator, and there will arise $y = \frac{a^3 z^{\frac{1}{2}n-1}}{a^2 + z^n}$, an Equation of the 1st Species of the 2d Order of Table 2, and the Terms being compared, it will be $d = a^3$, $e = a^2$, and $f = 1$; so that $\sqrt{\frac{a^3}{a^2 + z^2}} = x$, $\sqrt{a^3 - a^2 x^2} = v$, and $xv - 2s = t$.

87. Now that the values found of x and v may be reduced to a just number of dimensions, choose any given quantity, as a , by which, as unity, a^3 may be multiplied once in the value of x , and in the value of v , a^3 may be divided once, and $a^2 x^2$ twice. And by this means you will obtain $\sqrt{\frac{a^4}{a^2 + z^2}} = x$, $\sqrt{a^2 - x^2} = v$, and $xv - 2s = t$: of which the construction is thus.

88. Center A, and Radius AQ, describe the Quadrantal Arch QDP; in AC take $AB = AH$; raise the perpendicular BD meeting that Arch in D, and draw AD. Then the double of the Sector ADP will be equal to the Area sought ACEQ. For $\sqrt{\frac{a^4}{a^2 + z^2}} = (\sqrt{AD^2 - AB^2}) BD$, or v ; and $xv - 2s = 2\Delta ADB - 2ABDQ$, or $= 2\Delta ADB + 2BDP$, that is, either $= -2QAD$, or $= 2DAP$: Of which values the affirmative $2DAP$ belongs to the Area ACEQ on this side EC, and the negative $-2QAD$ belongs to the Area RECR extended *ad infinitum* beyond EC.

89. The solutions of Problems thus found may sometimes be made more elegant. Thus in the present case, drawing RH the semidiameter

90. EXAMPLE II. Let AGE be a Curve, which is described by the Angular point E of the *Norma* AEF, whilst one of the Legs AE, being interminate, passes continually through the given point A, and the other CE, of a given length,

$$\text{HA or } y = \frac{z^2}{\sqrt{a^2 - z^2}}.$$

or $2 = n$, and thence it will be $\frac{z^{\frac{3}{2}n-1}}{\sqrt{a^2-z^n}} = y$. Here since z in the numerator is of a fracted dimension, depress the value of y by di-

2d Species of the 7th Order of Table 2. And the terms being compared, it is $d = 1$, $e = -1$, and $f = a^2$. So that $z^2 =$

x and z are equal, and since $\sqrt{a^2 - x^2} = v$ is an Equation to a Circle, whose Diameter is a : with the Center A, and distance a or

Circle, whose Diameter is $2r$. Let the Circle PDQ be described, which CE meets in D , and let EF , let the Circle PDQ be described, which CE meets in D , and let the Parallelogram $ACDI$ be completed; then will $AC = z$, $CD = v$, and the Area sought $AGEC = s - xv = ACDP - ACDI = IDP$.

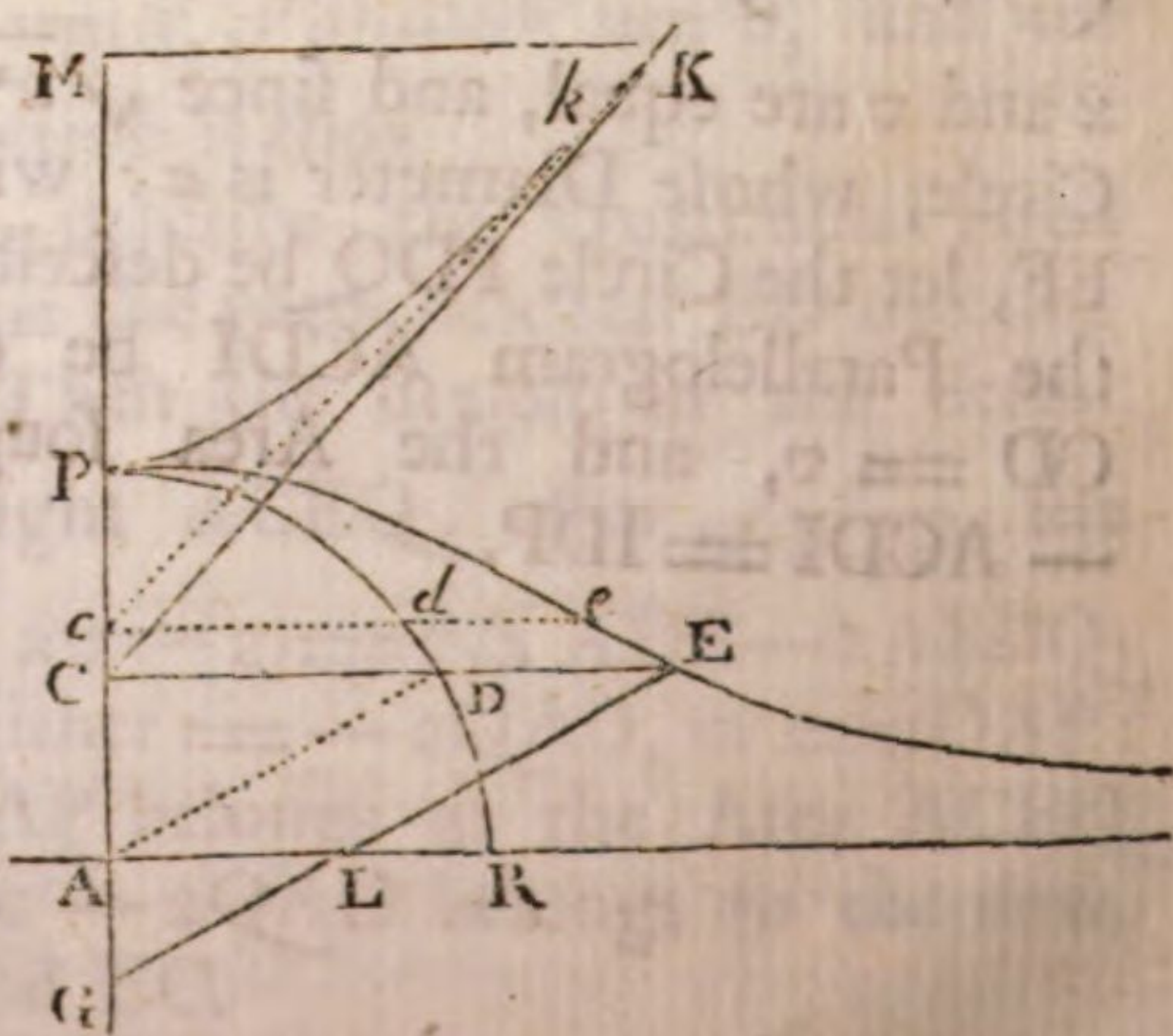
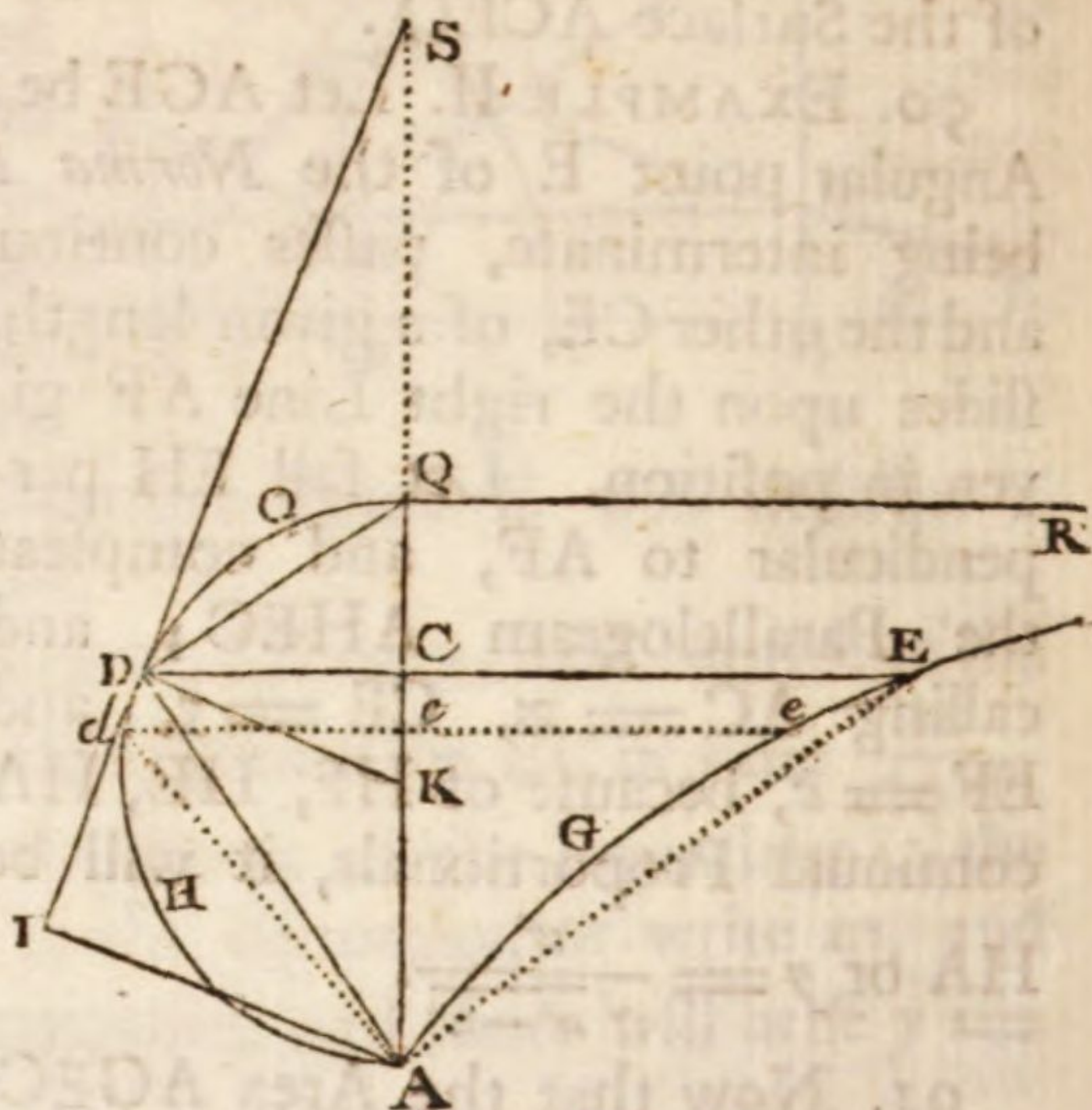
92. Ex-

92. EXAMPLE III. Let AGE be the Cissoïd belonging to the Circle ADQ, described with the diameter AQ. Let DCE be drawn perpendicular to the diameter, and meeting the Curves in D and E. And naming $AC = z$, $CE = y$, and $AQ = a$; because of CD, CA, CE continual Proportionals, it will be CE or $y = \frac{zz}{\sqrt{az - zz}}$, and dividing by z , 'tis

$$y = \frac{z}{\sqrt{az - 1}}. \text{ Therefore } z^{-1} = z^{\eta}, \text{ or } -1 = \eta, \text{ and thence } y = \frac{z^{-2\eta-1}}{\sqrt{az^{\eta}-1}}, \text{ an Equation of}$$

the 3d Species of the 4th Order of Table 2. The Terms therefore being compared, 'tis $d = 1$, $e = -1$, and $f = a$. Therefore $z = \frac{1}{z^{\eta}} = x$, $\sqrt{ax - xx} = v$, and $3s - 2xv = t$. Wherefore it is $AC = x$, $CD = v$, and thence $ACDH = s$; so that $3ACDH - 4\Delta ADC = 3s - 2xv = t = \text{Area of the Cissoïd ACEGA}$. Or, which is the same thing, 3 Segments ADHA = Area ADEGA, or 4 Segments ADHA = Area AHDEGA.

93. EXAMPLE IV. Let PE be the first Conchoid of the Ancients, described from Center G, with the Asymptote AL, and distance LE. Draw its Axis GAP, and let fall the Ordinate EC. Then calling $AC = z$, $CE = y$, $GA = b$, and $AP = c$; because of the Proportionals $AC : CE = AL : GC$, it will be CE or $y = \frac{b+z}{z} \sqrt{c^2 - z^2}$.



94. Now that its Area PEC may be found from hence, the parts of the Ordinate CE are to be consider'd separately. And if the Ordinate CE is so divided in D, that it is $CD = \sqrt{c^2 - z^2}$, and

let such a Circle PDQ be described, which CE produced meets in D; then it will be $DE = \frac{bz}{\sqrt{c^2 - z^2}}$: By the help of which Equation there remains the Area PDEP or DERQ to be determin'd. Suppose therefore $n=2$, and $\theta=b$, and it will be $DE = \frac{bz^{n-1}}{\sqrt{cc - z^n}}$, an Equation of the 1st Species of the 4th Order of Table I. And the Terms being compared, it will be $b=d$, $cc=e$, and $-1=f$; so that $-b\sqrt{cc - zz} = -bR = t$.

98. Now as the value of t is negative, and therefore the Area represented by t lies beyond the Line DE; that its initial Limit may be found, seek for that length of z , at which t becomes nothing, and you will find it to be c . Therefore continue AC to Q, that it may be $AQ=cc$, and erect the Ordinate QR; and DQRED will be the Area whose value now found is $-b\sqrt{cc - zz}$.

99. If you should desire to know the quantity of the Area PDE, posited at the Absciss AC, and co-extended with it, without knowing the Limit QR, you may thus determine it.

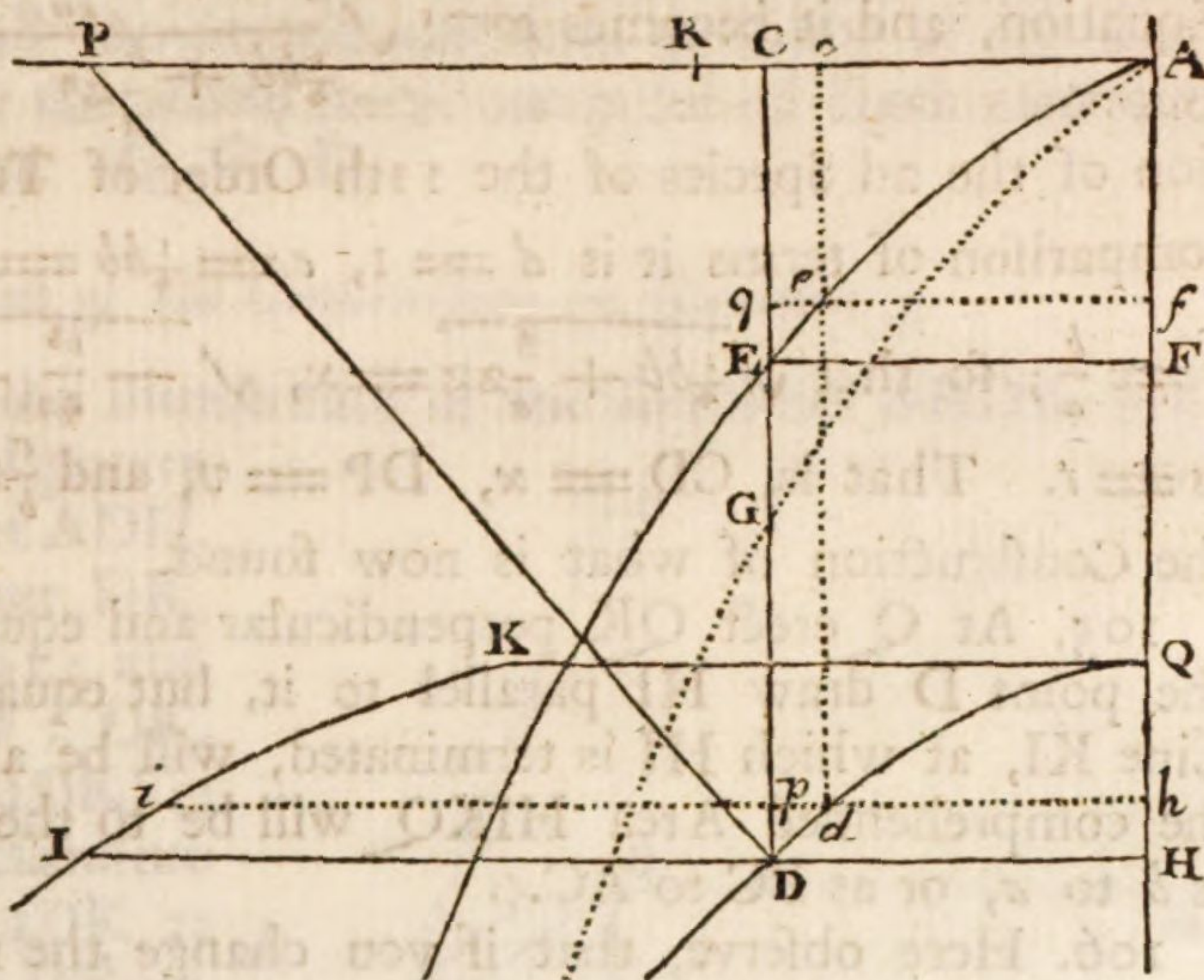
100. From the Value which t obtains at the length of the Absciss AC, subtract its value at the beginning of the Absciss; that is, from $-b\sqrt{cc - zz}$ subtract $-bc$, and there will arise the desired quantity $bc - b\sqrt{cc - zz}$. Therefore compleat the Parallelogram PAGK, and let fall DM perpendicular to AP, which meets GK in M; and the Parallelogram PKML will be equal to the Area PDE.

101. Whenever the Equation defining the nature of the Curve cannot be found in the Tables, nor can be reduced to simpler terms by division, nor by any other means; it must be transform'd into other Equations of Curves related to it, in the manner shewn in Prob. 8. till at last one is produced, whose Area may be known by the Tables. And when all endeavours are used, and yet no such can be found, it may be certainly concluded, that the Curve proposed cannot be compared, either with rectilinear Figures, or with the Conic Sections.

102. In the same manner when mechanical Curves are concern'd, they must first be transform'd into equal Geometrical Figures, as is shewn in the same Prob. 8. and then the Areas of such Geometrical Curves are to be found from the Tables. Of this matter take the following Example.

103. EXAMPLE 6. Let it be proposed to determine the Area of the Figure of the Arches of any Conic Section, when they are made Ordinates on their Right Sines. As let A be the Center of the Conic Section,

AQ and AR the Semiaxes, CD the Ordinate to the Axis AR, and PD a Perpendicular at the point D. Also let AE be the said mechanical Curve meeting CD in E; and from its nature before defined, CE will be equal to the Arch QD. Therefore the Area AEC is sought, or completing the parallelogram ACEF, the excess AEF is required. To which purpose let a be the Latus rectum of the Conic Section, and b its Latus transversum, or $2AQ$. Also let $AC=z$, and $CD=y$;



then it will be $\sqrt{\frac{1}{4}bb + \frac{b}{a}zz} = y$, an Equation to a Conic Section, as is known. Also $PC = \frac{b}{a}z$, and thence $PD = \sqrt{\frac{1}{4}bb + \frac{bb+ab}{aa}zz}$.

104. Now since the fluxion of the Arch QD is to the fluxion of the Absciss AC, as PD to CD; if the fluxion of the Absciss be suppos'd 1, the Fluxion of the Arch QD, or of the Ordinate CE,

will be $\sqrt{\frac{\frac{1}{4}bb + \frac{bb+ab}{aa}zz}{\frac{1}{4}bb + \frac{b}{a}zz}}$. Draw this into FE, or z , and there

will arise $z \sqrt{\frac{\frac{1}{4}bb + \frac{bb+ab}{aa}zz}{\frac{1}{4}bb + \frac{b}{a}zz}}$ for the fluxion of the Area AEF.

If therefore in the Ordinate CD you take $CG = z \sqrt{\frac{\frac{1}{4}bb + \frac{bb+ab}{aa}zz}{\frac{1}{4}bb + \frac{b}{a}zz}}$, the Area AGC, which is described by CG

moving upon AC, will be equal to the Area AEF, and the Curve

Q²

AG

AG will be a Geometrical Curve. Therefore the Area AGC is sought. To this purpose let z^n be substituted for z^2 in the last

Equation, and it becomes $z^{n-1} \sqrt{\frac{\frac{1}{4}bb + \frac{bb+ab}{aa}z^n}{\frac{1}{4}bb + \frac{b}{a}z^n}} = CG$, an Equa-

tion of the 2d Species of the 11th Order of Table 2. And from a comparison of terms it is $d = 1$, $e = \frac{1}{4}bb = g$, $f = \frac{bb+ab}{aa}$, and $b = \frac{b}{a}$; so that $\sqrt{\frac{1}{4}bb + \frac{b}{a}zz} = x$, $\sqrt{-\frac{b^2}{4a} + \frac{a+b}{a}xx} = v$, and $\frac{a}{b}s = t$. That is, $CD = x$, $DP = v$, and $\frac{a}{b}s = t$. And this is the Construction of what is now found.

105. At Q erect QK perpendicular and equal to QA, and thro' the point D draw HI parallel to it, but equal to DP. And the Line KI, at which HI is terminated, will be a Conic Section, and the comprehended Area HIKQ will be to the Area sought AEF, as b to a , or as PC to AC.

106. Here observe, that if you change the sign of b , the Conic Section, to whose Arch the right Line CE is equal, will become an Ellipsis; and besides, if you make $b = -a$, the Ellipsis becomes a Circle. And in this case the line KI becomes a right line parallel to AQ.

107. After the Area of any Curve has been thus found and constructed, we should consider about the demonstration of the construction; that laying aside all Algebraical calculation, as much as may be, the Theorem may be adorn'd, and made elegant, so as to become fit for publick view. And there is a general method of demonstrating, which I shall endeavour to illustrate by the following Examples.

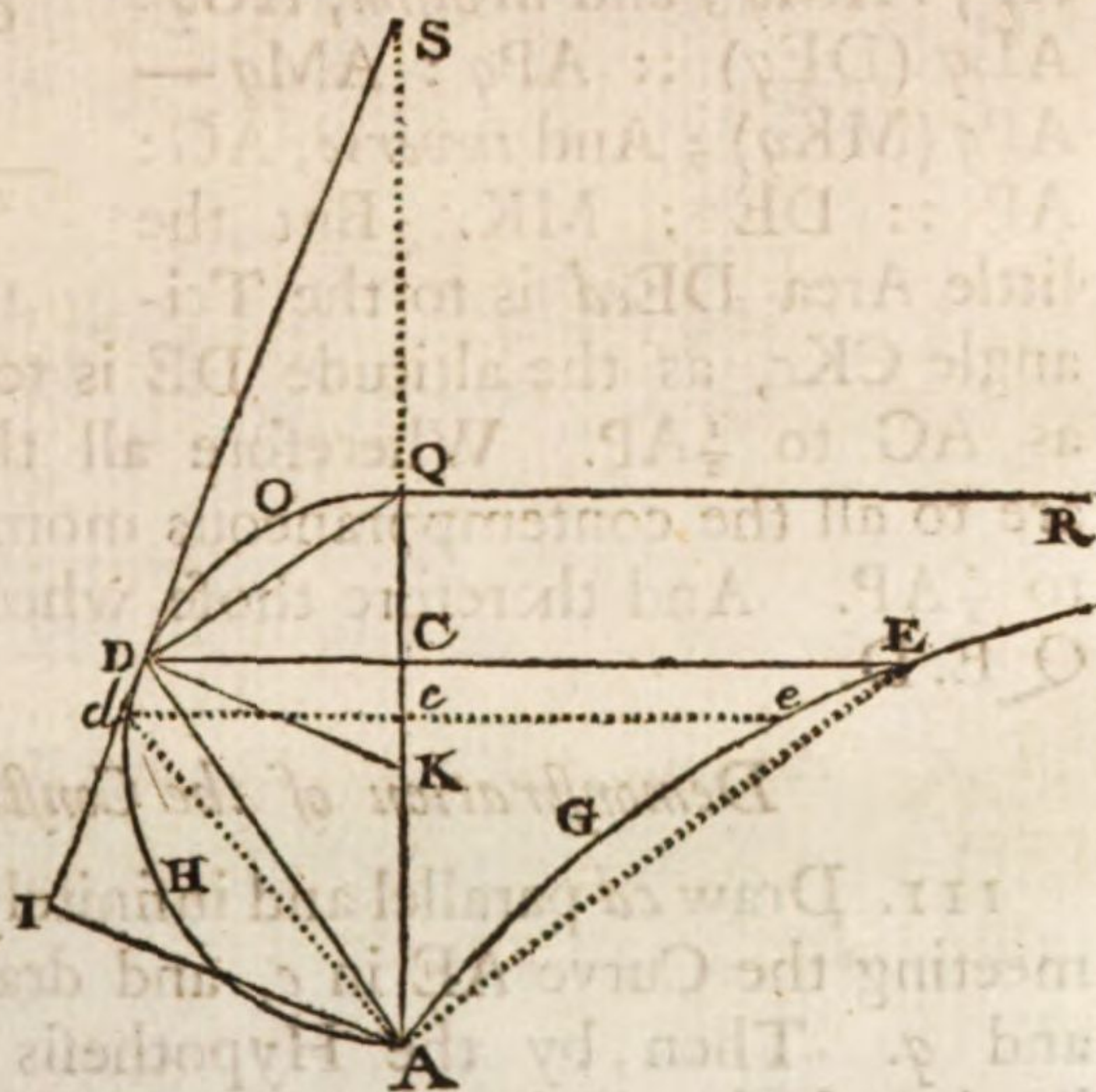
Demonstration of the Construction in Example 5.

108. In the Arch PQ take a point d indefinitely near to D, (Figure p. 113.) and draw de and dm parallel to DE and DM, meeting DM and AP in p and l . Then will DEed be the moment of the Area PDEP, and LMml will be the moment of the Area LMKP. Draw the semidiameter AD, and conceive the indefinitely small arch Dd to be as it were a right line, and the triangles Dpd and ALD will be like, and therefore $Dp : pd :: AL : LD$. But it is $HF : EH :: AG : AF$; that is, $AL : LD :: ML : DE$; and therefore $Dp : pd :: ML : DE$. Wherefore $Dp \times DE = pd \times ML$.
That

That is, the moment $DEed$ is equal to the moment $LMml$. And since this is demonstrated indeterminately of any contemporaneous moments whatever, it is plain, that all the moments of the Area $PDEP$ are equal to all the contemporaneous moments of the Area $PLMK$, and therefore the whole Areas composed of those moments are equal to each other. Q. E. D.

Demonstration of the Construction in Example 3.

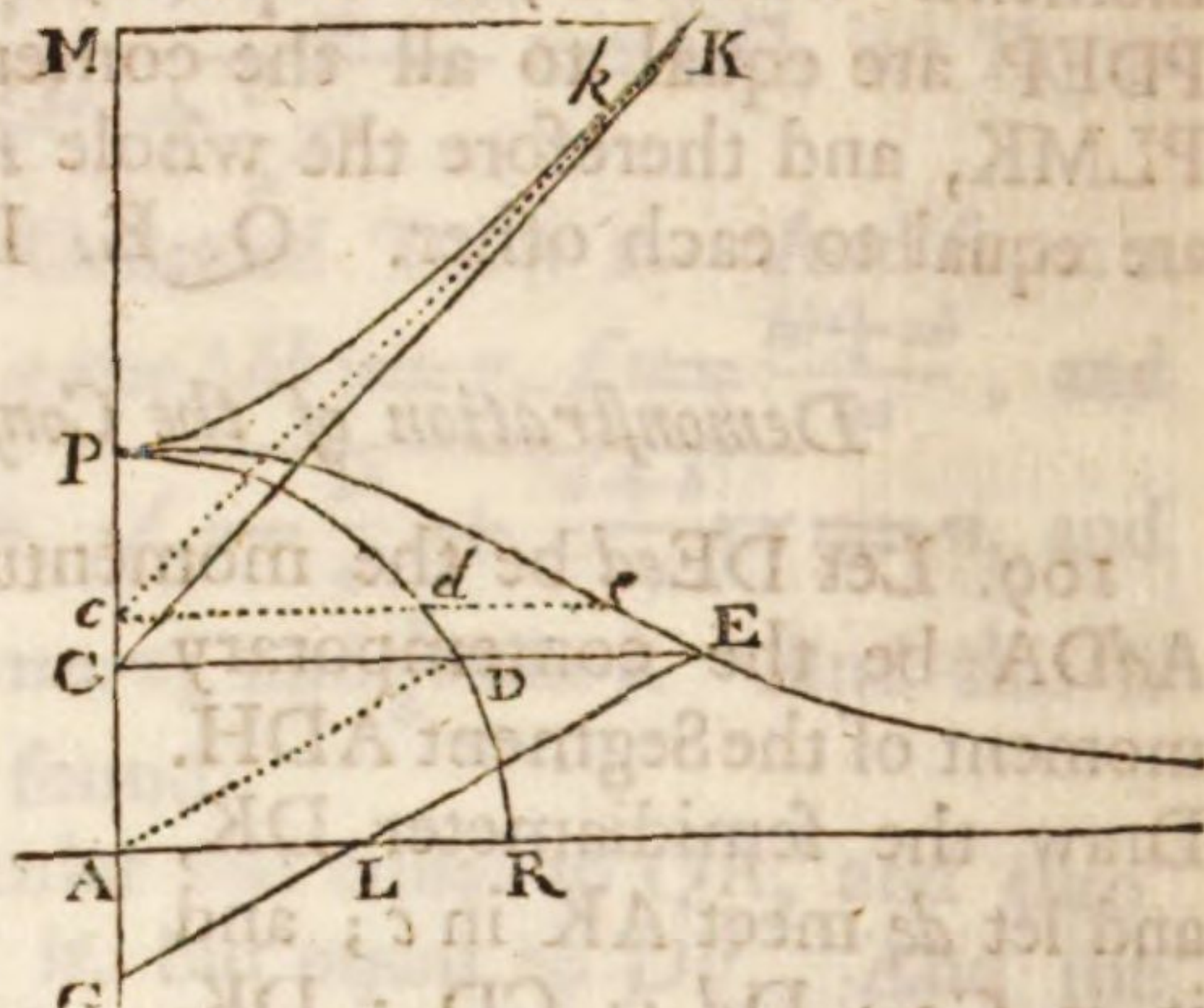
109. Let $DEed$ be the momentum of the superficies $AHDE$, and $AdDA$ be the contemporary moment of the Segment ADH . Draw the semidiameter DK , and let de meet AK in c ; and it is $Cc : Dd :: CD : DK$. Besides it is $DC : QA (2DK) :: AC : DE$. And therefore $Cc : 2Dd :: DC : 2DK :: AC : DE$, and $Cc \times DE = 2Dd \times AC$. Now to the moment of the periphery Dd produced, that is, to the tangent of the Circle, let fall the perpendicular AI , and AI will be equal to AC . So that $2Dd \times AC = 2Dd \times AI = 4$ Triangles ADd . So that 4 Triangles $ADd = Cc \times DE =$ moment $DEed$. Therefore every moment of the space $AHDE$ is quadruple of the contemporary moment of the Segment ADH , and therefore that whole space is quadruple of the whole Segment. Q. E. D.



Demonstration

Demonstration of the Construction in Example 4.

110. Draw ce parallel to CE , and at an indefinitely small distance from it, and the tangent of the Hyperbola ck , and let fall KM perpendicular to AP . Now from the nature of the Hyperbola it will be $AC : AP :: AP : AM$, and therefore $AGq : GLq :: ACq : LEq$ (or APq) :: $APq : AMq$; and *divisim*, $AGq : ALq$ (DEq) :: $APq : AMq$ — APq (MKq); And *inverse*, $AG : AP :: DE : MK$. But the little Area $DEed$ is to the Triangle CKc , as the altitude DE is to half the altitude KM ; that is, as AG to $\frac{1}{2}AP$. Wherefore all the moments of the Space PDE are to all the contemporaneous moments of the Space PKC , as AG to $\frac{1}{2}AP$. And therefore those whole Spaces are in the same ratio. Q. E. D.

*Demonstration of the Construction in Example 6.*

111. Draw cd parallel and infinitely near to CD , (Fig. in p. 115.) meeting the Curve AE in e , and draw bi and fe meeting DC in p and q . Then by the Hypothesis $Dd = Eq$, and from the similitude of the Triangles Ddp and DCP , it will be $Dp : (Dd) Eq :: (P : (PD) HI$, so that $Dp \times HI = Eq \times CP$; and thence $Dp \times HI$ (the moment $HIib$) : $Eq \times AC$ (the moment $EFfe$) :: $Eq \times CP : Eq \times AC :: CP : AC$. Wherefore since PC and AC are in the given ratio of the latus transversum to the latus rectum of the Conic Section QD , and since the moments $HIib$ and $EFfe$ of the Areas $HIKQ$ and AEF are in that ratio, the Areas themselves will be in the same ratio. Q. E. D.

112. In this kind of demonstrations it is to be observed, that I assume such quantities for equal, whose ratio is that of equality: And that is to be esteem'd a ratio of equality, which differs less from equality than by any unequal ratio that can be assign'd. Thus in the last demonstration I suppos'd the rectangle $Eq \times AC$, or $FEqf$, to be equal to the space $FEef$, because (by reason of the difference Ege infinitely less than them, or nothing in comparison of them,) they

they have not a ratio of inequality. And for the same reason I made $DP \times HI = HIib$; and so in others.

113. I have here made use of this method of proving the Areas of Curves to be equal, or to have a given ratio, by the equality, or by the given ratio, of their moments; because it has an affinity to the usual methods in these matters. But that seems more natural which depends upon the generation of Superficies, by Motion or Fluxion. Thus if the Construction in Example 2. was to be demonstrated: From the nature of the Circle, the fluxion of the right line ID (Fig. p. 111.) is to the fluxion of the right line IP, as AI to ID; and it is $AI : ID :: ID : CE$, from the nature of the Curve

AGE; and therefore $CE \times \dot{ID} = ID \times \dot{IP}$. But $CE \times \dot{ID} =$ to the fluxion of the Area PDI. And therefore those Areas, being generated by equal fluxion, must be equal. Q. E. D.

114. For the sake of farther illustration, I shall add the demonstration of the Construction, by which the Area of the Cissoïd is determin'd, in Example 3. Let the lines mark'd with points in the scheme be expunged; draw the Chord DQ , and the Asymptote QR of the Cissoïd. Then, from the nature of the Circle, it is

$DQq = AQ \times CQ$, and
thence (by Prob. 1.) $2DQ \times$

$$\text{Fluxion of } DQ = AQ \times \dot{CQ}.$$

And therefore $AQ : DQ ::$

$2\dot{DQ} : \dot{CQ}$. Also from the

nature of the Ciffoid it is ED :
AD :: AO : DO. There-

fore $ED : AD :: {}^2D\ddot{O} : \ddot{C}O$.

$$\text{and } ED \times \dot{CO} = AD \times \dot{DO}.$$

or $\frac{1}{2} \times \frac{1}{2} AD \times DO$. Now since

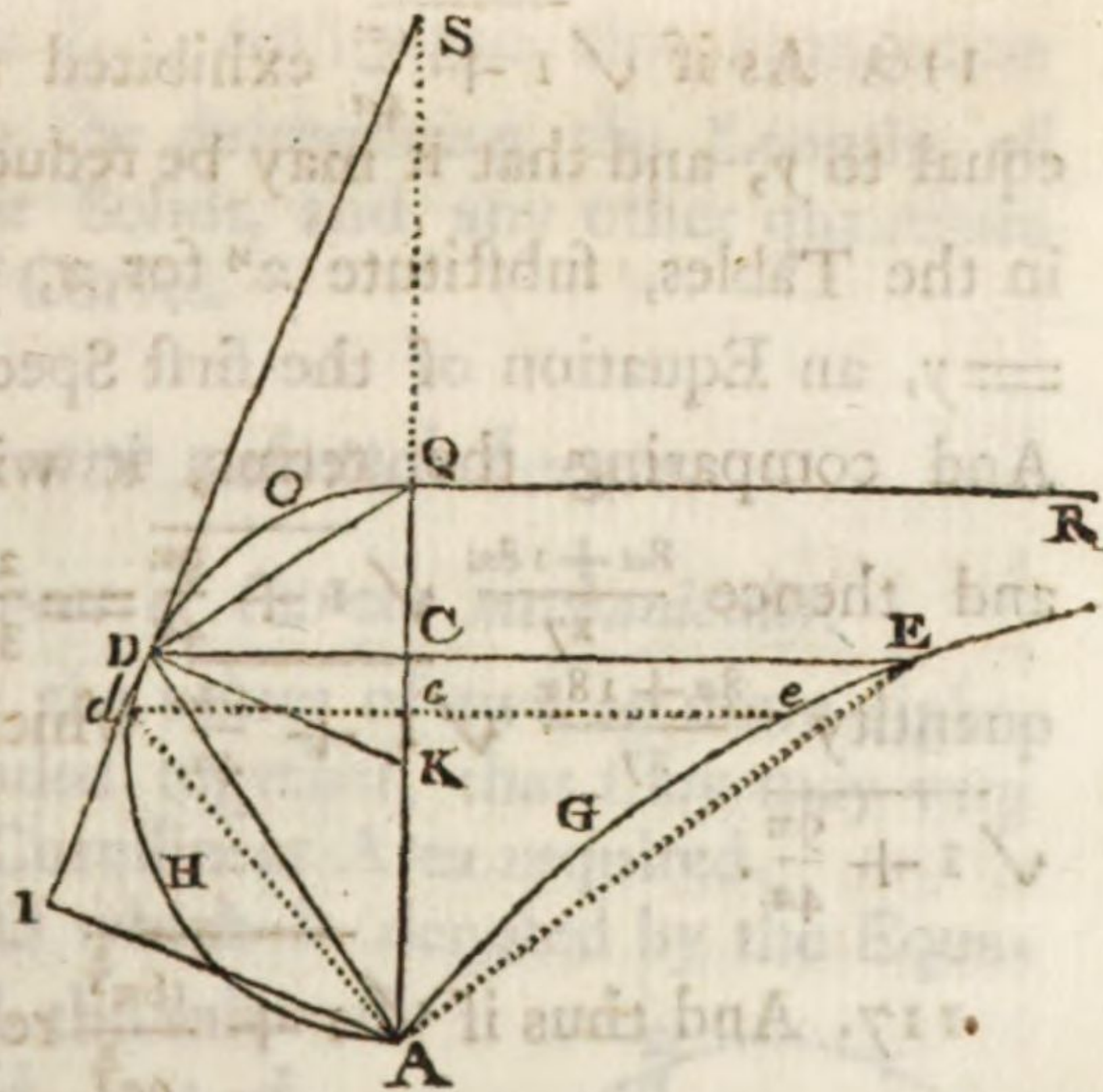
DQ is perpendicular at the

end of AD, revolving about

A; and $\frac{1}{2}AD \times QD =$ to the

its quadruple also $ED \times CQ =$
OREDO. Wherefore that A

rated quadruple of the other



A ; and $\frac{r}{2}AD \times \dot{QD} =$ to the

its quadruple also $ED \times \dot{CQ} =$

QREDO. Wherefore that A
rated quadruple of the other

...then their values are to be

SCHOLIUM.

SCHOLIUM.

115. By the foregoing Tables not only the Areas of Curves, but quantities of any other kind, that are generated by an analogous way of flowing, may be derived from their Fluxions, and that by the assistance of this Theorem: That a quantity of any kind is to an unit of the same kind, as the Area of a Curve is to a superficial unity; if so be that the fluxion generating that quantity be to an unit of its kind, as the fluxion generating the Area is to an unit of its kind also; that is, as the right Line moving perpendicularly upon the Absciss (or the Ordinate) by which the Area is described, to a linear Unit. Wherefore if any fluxion whatever is expounded by such a moving Ordinate, the quantity generated by that fluxion will be expounded by the Area described by such Ordinate; or if the Fluxion be expounded by the same Algebraic terms as the Ordinate, the generated quantity will be expounded by the same as the described Area. Therefore the Equation, which exhibits a Fluxion of any kind, is to be sought for in the first Column of the Tables, and the value of t in the last Column will show the generated Quantity.

116. As if $\sqrt{1 + \frac{9z}{4a}}$ exhibited a Fluxion of any kind, make it equal to y , and that it may be reduced to the form of the Equations in the Tables, substitute z^n for z , and it will be $z^{n-1} \sqrt{1 + \frac{9z^n}{4a}} = y$, an Equation of the first Species of the 3d Order of Table I. And comparing the terms, it will be $d = 1$, $e = 1$, $f = \frac{9}{4a}$, and thence $\frac{8a+18z}{27} \sqrt{1 + \frac{9z}{4a}} = \frac{2d}{3nf} R^3 = t$. Therefore it is the quantity $\frac{8a+18z}{27} \sqrt{1 + \frac{9z}{4a}}$ which is generated by the Fluxion $\sqrt{1 + \frac{9z}{4a}}$.

117. And thus if $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ represents a Fluxion, by a due reduction, (or by extracting $z^{\frac{2}{3}}$ out of the radical, and writing z^n for $z^{-\frac{2}{3}}$;) there will be had $\frac{1}{z^{\frac{1}{3}+1}} \sqrt{z^n + \frac{16}{9a^{\frac{2}{3}}}} = y$, an Equation of the 2d Species of the 5th Order of Table 2. Then comparing the terms,

terms, it is $d = 1$, $e = \frac{16}{9a^{\frac{2}{3}}}$, and $f = 1$. So that $z^{\frac{2}{3}} = \frac{1}{x^4} = xx$,

$\sqrt{1 + \frac{16xx}{9a^{\frac{2}{3}}}} = v$, and $\frac{3}{2}s = \frac{-2d}{n}s = t$. Which being found, the

quantity generated by the fluxion $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ will be known, by

making it to be to an Unit of its own kind, as the Area $\frac{3}{2}s$ is to superficial unity; or which comes to the same, by supposing the quantity t no longer to represent a Superficies, but a quantity of another kind, which is to an unit of its own kind, as that superficies is to superficial unity.

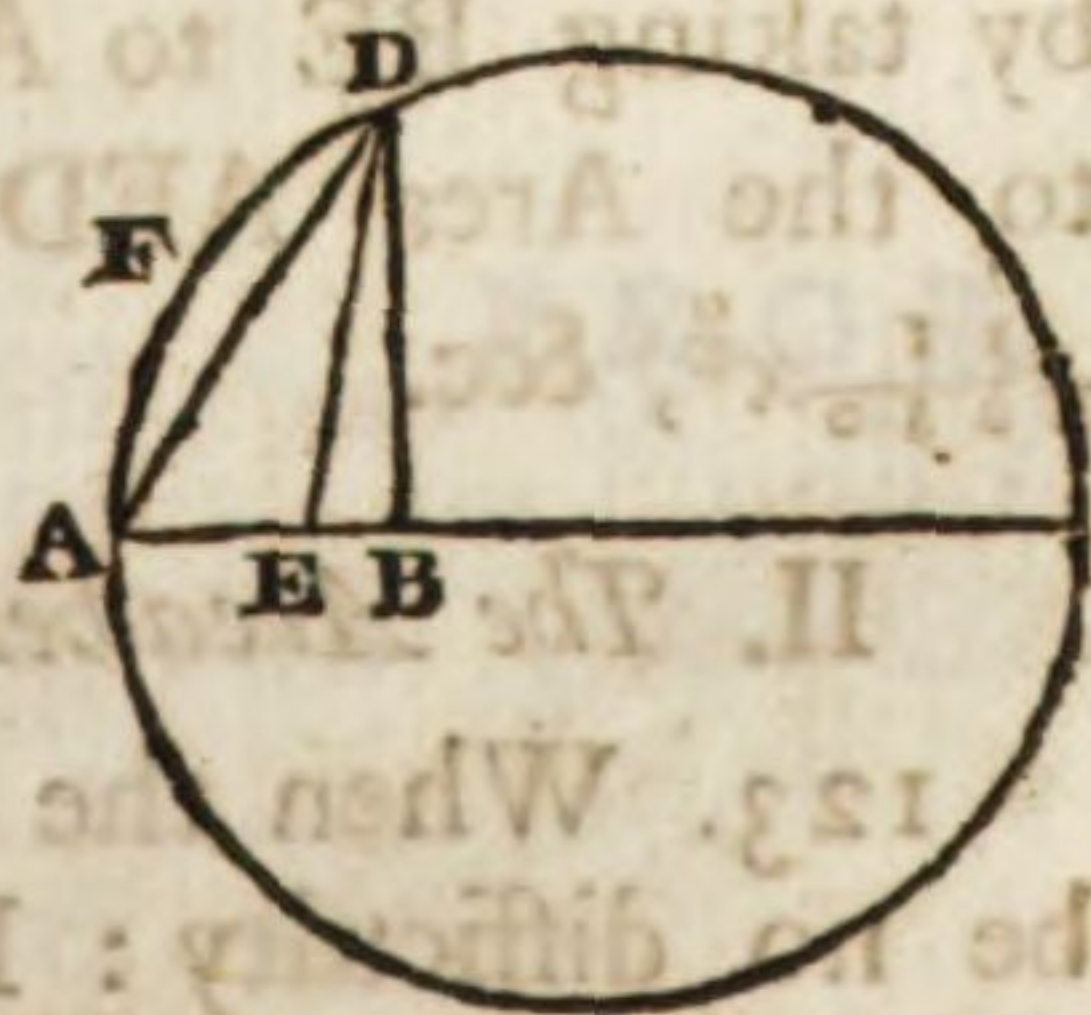
118. Thus supposing $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}}$ to represent a linear Fluxion, I imagine t no longer to signify a Superficies, but a Line; that Line, for instance, which is to a linear unit, as the Area which (according to the Tables) is represented by t , is to a superficial unit, or that which is produced by applying that Area to a linear unit. On which account, if that linear unit be made e , the length generated by the foregoing fluxion will be $\frac{3s}{2e}$. And upon this foundation those Tables may be apply'd to the determining the Lengths of Curve-lines, the Contents of their Solids, and any other quantities whatever, as well as the Areas of Curves.

Of Questions that are related hereto.

I. To approximate to the Areas of Curves mechanically.

119. The method is this, that the values of two or more right-lined Figures may be so compounded together, that they may very nearly constitute the value of the Curvilinear Area required.

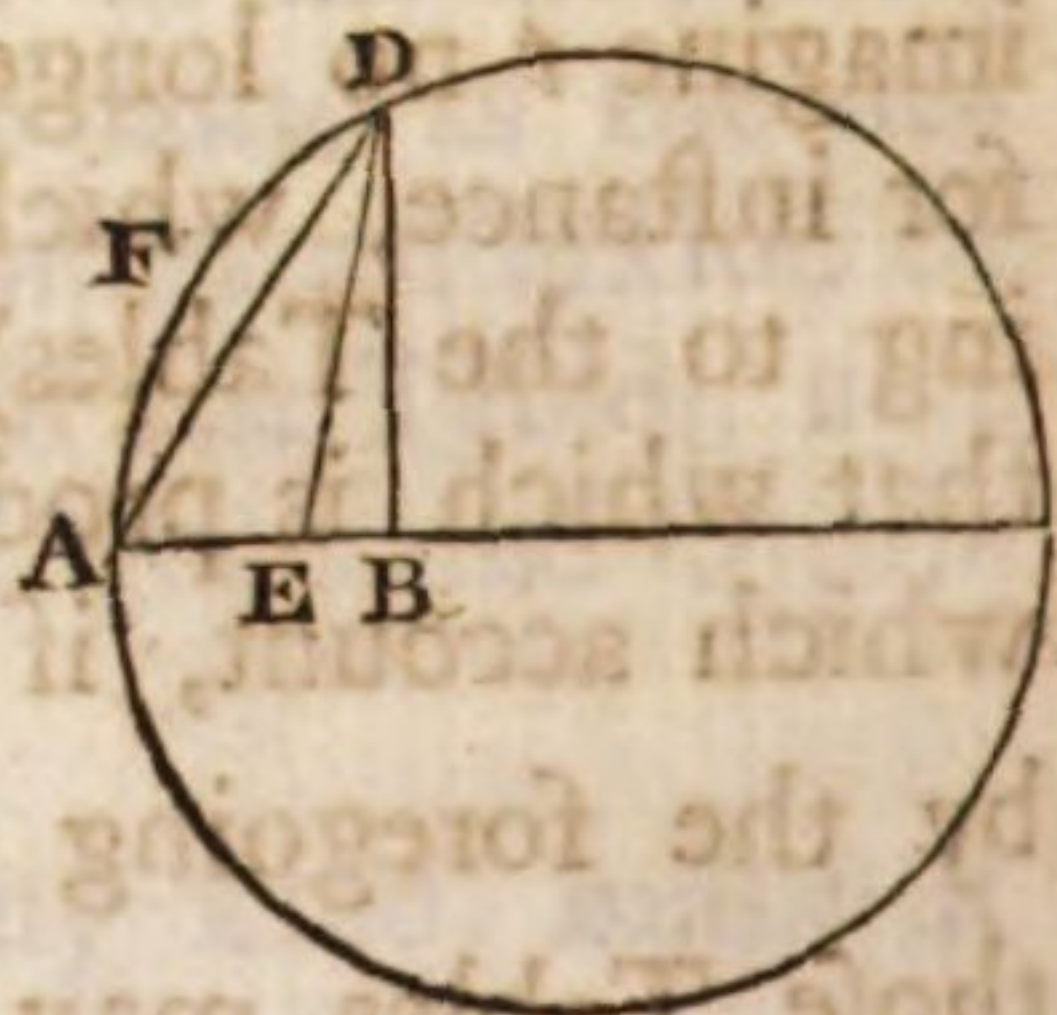
120. Thus for the Circle AFD which is denoted by the Equation $x - xx = zz$, having found the value of the Area AFDB, viz. $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$, &c. the values of some Rectangles are to be sought, such is the value $x\sqrt{x - xx}$, or $x^{\frac{3}{2}} - \frac{1}{2}x^{\frac{5}{2}} - \frac{1}{8}x^{\frac{7}{2}} - \frac{1}{16}x^{\frac{9}{2}}$, &c. of the rectangle $BD \times AB$, and $x\sqrt{x}$, or $x^{\frac{3}{2}}$, the value of $AD \times AB$. Then these values are to be multiply'd by



to be added together, and the terms of the sum are to be compared with the corresponding terms of the value of the Area AFDB, that as far as is possible they may become equal. As if those Parallelograms were multiply'd by e and f , the sum would be $ex^{\frac{3}{2}} - \frac{1}{2}ex^{\frac{5}{2}} + f$

$-\frac{1}{8}ex^{\frac{7}{2}}$, &c. the terms of which being compared with these terms $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}}$, &c. there arises $e + f = \frac{2}{3}$, and $-\frac{1}{2}e = -\frac{1}{5}$, or $e = \frac{2}{5}$, and $f = \frac{2}{3} - e = \frac{4}{15}$. So that $\frac{2}{5}BD \times AB + \frac{4}{15}AD \times AB = \text{Area AFDB}$ very nearly. For $\frac{2}{5}BD \times AB + \frac{4}{15}AD \times AB$ is equivalent to $\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{40}x^{\frac{9}{2}}$, &c. which being subtracted from the Area AFDB, leaves the error only $\frac{1}{70}x^{\frac{7}{2}} + \frac{1}{90}x^{\frac{9}{2}}$, &c.

121. Thus if AB were bisected in E, the value of the rectangle $AB \times DE$ will be $x\sqrt{x} - \frac{3}{4}xx$, or $x^{\frac{3}{2}} - \frac{3}{8}x^{\frac{5}{2}} - \frac{9}{128}x^{\frac{7}{2}} - \frac{27}{1024}x^{\frac{9}{2}}$, &c. And this compared with the rectangle $AD \times AB$, gives $\frac{8DE + 2AD}{15}$ into $AB = \text{Area AFDB}$, the error being only $\frac{1}{560}x^{\frac{3}{2}} + \frac{1}{5760}x^{\frac{5}{2}}$, &c. which is always less than $\frac{1}{1500}$ part of the whole Area, even tho' AFDB were a quadrant of a Circle. But this Theorem may be thus propounded. As 3 to 2, so is the rectangle AB into DE, added to a fifth part of the difference between AD and DE, to the Area AFDB, very nearly.



122. And thus by compounding two rectangles $AB \times ED$ and $AB \times BD$, or all the three rectangles together, or by taking in still more rectangles, other Rules may be invented, which will be so much the more exact, as there are more Rectangles made use of. And the same is to be understood of the Area of the Hyperbola, or of any other Curves. Nay, by one only rectangle the Area may often be very commodiously exhibited, as in the foregoing Circle, by taking BE to AB as $\sqrt{10}$ to 5, the rectangle $AB \times ED$ will be to the Area AFDB, as 3 to 2, the error being only $\frac{1}{175}x^{\frac{3}{2}} + \frac{1}{3150}x^{\frac{5}{2}}$, &c.

II. The Area being given, to determine the Absciss and Ordinate.

123. When the Area is express'd by a finite Equation, there can be no difficulty: But when it is express'd by an infinite Series, the affected root is to be extracted, which denotes the Absciss. So for the

the Hyperbola, defined by the Equation $\frac{ab}{a+x} = z$, after we have found $z = bx - \frac{bx^2}{2a} + \frac{bx^3}{3a^2} - \frac{bx^4}{4a^3}$, &c. that from the given Area the Absciss x may be known, extract the affected Root, and there will arise $x = \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{96a^4b^5}$, &c. And moreover, if the Ordinate z were required, divide ab by $a+x$, that is, by $a + \frac{z}{b} + \frac{z^2}{2ab^2} + \frac{z^3}{6a^2b^3}$, &c. and there will arise $z = b - \frac{z}{a} - \frac{z^2}{2a^2b} - \frac{z^3}{6a^3b^2} - \frac{z^4}{24a^4b^3}$, &c.

124. Thus as to the Ellipsis which is express'd by the Equation $ax - \frac{a}{c}xx = zz$, after the Area is found $z = \frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{a^{\frac{1}{2}}x^{\frac{5}{2}}}{5c} - \frac{a^{\frac{1}{2}}x^{\frac{7}{2}}}{28c^2} - \frac{a^{\frac{1}{2}}x^{\frac{9}{2}}}{72c^3}$, &c. write v^3 for $\frac{3z}{2a^{\frac{1}{2}}}$, and t for $x^{\frac{1}{2}}$, and it becomes $v^3 = t^3 - \frac{3t^5}{10c} - \frac{3t^7}{56c^2} - \frac{t^9}{48c^3}$, &c. and extracting the root $t = v + \frac{v^3}{10c} + \frac{81v^5}{1400c^2} + \frac{1171v^7}{25200c^3}$, &c. whose square $v^2 + \frac{v^4}{5c} + \frac{22v^6}{175c^2} + \frac{823v^8}{7875c^3}$, &c. is equal to x . And this value being substituted instead of x in the Equation $ax - \frac{a}{c}xx = zz$, and the root being extracted, there arises $z = a^{\frac{1}{2}}v - \frac{2a^{\frac{1}{2}}v^3}{5c} - \frac{38a^{\frac{1}{2}}v^5}{175c^2} - \frac{407a^{\frac{1}{2}}v^7}{2250c^3}$, &c. So that from z , the given Area, and thence v or $\sqrt[3]{\frac{3z}{2a^{\frac{1}{2}}}}$, the Absciss x will be given, and the Ordinate z . All which things may be accommodated to the Hyperbola, if only the sign of the quantity c be changed, wherever it is found of odd dimensions.

P R O B. X.

To find as many Curves as we please, whose Lengths may be express'd by finite Equations.

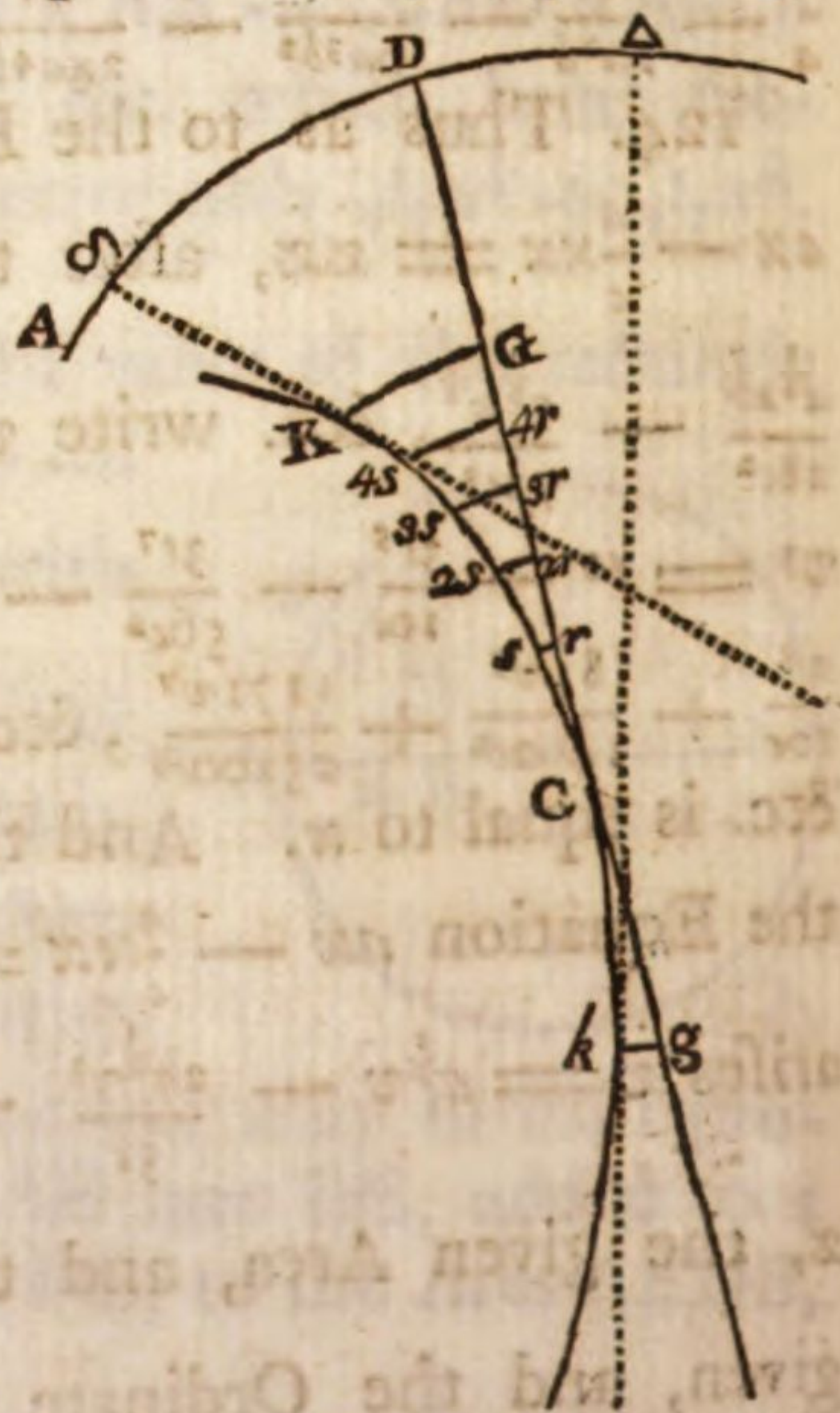
1. The following positions prepare the way for the solution of this Problem.

2. I. If the right Line DC, standing perpendicularly upon any Curve AD, be conceived thus to move, all its points G, g, r, &c. will describe other Curves, which are equidistant, and perpendicular to that line: As GK, gk, rs, &c.

3. II. If that right Line is continued indefinitely each way, its extremities will move contrary ways, and therefore there will be a Point between, which will have no motion, but may therefore be call'd the Center of Motion. This Point will be the same as the Center of Curvature, which the Curve AD hath at the point D, as is mention'd before. Let that point be C.

4. III. If we suppose the line AD not to be circular, but unequally curved, suppose more curved towards δ , and less toward Δ ; that Center will continually change its place, approaching nearer to the parts more curved, as in K, and going farther off at the parts less curved, as in k, and by that means will describe some line, as KCk.

5. IV. The right Line DC will continually touch the line described by the Center of Curvature. For if the Point D of this line moves towards δ , its point G, which in the mean time passes to K, and is situate on the same side of the Center C, will move the same way, by position 2. Again, if the same point D moves towards Δ , the point g, which in the mean time passes to k, and is situate on the contrary side of the Center C, will move the contrary way, that is, the same way that G moved in the former case, while it pass'd to K. Wherefore K and k lie on the same side of the right Line DC. But as K and k are taken indefinitely for any points,



points, it is plain that the whole Curve lies on the same side of the right line DC, and therefore is not cut, but only touch'd by it.

6. Here it is suppos'd, that the line $\delta D \Delta$ is continually more curved towards δ , and less towards Δ ; for if its greatest or least Curvature is in D, then the right line DC will cut the Curve KC; but yet in an angle that is less than any right-lined angle, which is the same thing as if it were said to touch it. Nay, the point C in this case is the Limit, or Cuspid, at which the two parts of the Curve, finishing in the most oblique concourse, touch each other; and therefore may more justly be said to be touch'd, than to be cut, by the right line DC, which divides the Angle of contact.

7. V. The right Line CG is equal to the Curve CK. For conceive all the points $r, 2r, 3r, 4r, \&c.$ of that right Line to describe the arches of Curves $rs, 2r2s, 3r3s, \&c.$ in the mean time that they approach to the Curve CK, by the motion of that right line; and since those arches, (by position 1.) are perpendicular to the right lines that touch the Curve CK, (by position 4.) it follows that they will be also perpendicular to that Curve. Wherefore the parts of the line CK, intercepted between those arches, which by reason of their infinite smallness may be consider'd as right lines, are equal to the intervals of the same arches; that is, (by position 1.) are equal to so many parts of the right line CG. And equals being added to equals, the whole Line CK will be equal to the whole Line CG.

8. The same thing would appear by conceiving, that every part of the right Line CG, as it moves along, will apply itself successively to every part of the Curve CK, and thereby will measure them; just as the Circumference of a wheel, as it moves forward by revolving upon a Plain, will measure the distance that the point of Contact continually describes.

9. And hence it appears, that the Problem may be resolved, by assuming any Curve at pleasure $A\delta D \Delta$, and thence by determining the other Curve KCk , in which the Center of Curvature of the assumed Curve is always found. Therefore letting fall the perpendiculars DB and CL, to a right Line AB given in position, and in AB taking any point A, and calling $AB = x$ and $BD = y$; to define the Curve AD let any relation be assumed between x and y , and then by Prob. 5. the point C may be found, by which may be determined both the Curve KC, and its Length GC.

it will be $\Delta e : \Delta f :: \overline{\Delta e}^2 : \overline{\Delta f}^2 :: \overline{Cg}^2 : \overline{C\Delta}^2 :: \frac{\overline{Cg}^2}{C\Delta} : C\Delta$.

And $\Delta f : Pp :: C\Delta : CP$. Then *ex æquo*, $\Delta e : Pp :: \frac{\overline{Cg}^2}{C\Delta} : CP$.

But Pp is the moment of the Absciss AP , by the accession of which it becomes Ap ; and Δe is the contemporaneous moment of the perpendicular Δg , by the decrease of which it becomes δg . Therefore Δe and Pp are as the fluxions of the lines Δg (z) and AP (x),

that is, as $\dot{z}$ and $\dot{x}$. Wherefore $\dot{z} : \dot{x} :: \frac{\overline{Cg}^2}{C\Delta} : CP$. And since it

is $\overline{Cg}^2 = \overline{C\Delta}^2 - \overline{\Delta g}^2 = 1 - zz$, and $C\Delta = 1$; it will be

$CP = \frac{x - xz^2}{z}$. Moreover since we may assume any one of the

three $\dot{x}$, $\dot{y}$, and $\dot{z}$ for an uniform fluxion, to which the rest are to be referr'd, if $\dot{x}$ be that fluxion, and its value is unity, then $CP =$

$\frac{1 - zz}{z}$.

13. Besides it is $C\Delta (1) : \Delta g (z) :: CP : PL$; also $C\Delta (1) : Cg (\sqrt{1 - zz}) :: CP : CL$; therefore it is $PL = \frac{z - z^3}{z}$, and $CL =$

$\frac{1 - zz}{z} \sqrt{1 - zz}$. Lastly, drawing pq parallel to the infinitely small

Arch Dd , or perpendicular to DC , Pq will be the momentum of DP , by the accession of which it becomes dp , at the same time that AP becomes Ap . Therefore Pp and Pq are as the fluxions of AP (x) and PD (y), that is, as 1 and $\dot{y}$. Therefore because of similar triangles Ppq and $C\Delta g$, since $C\Delta$ and Δg , or 1 and z , are in the same ratio, it will be $y = z$. Whence we have this solution of the Problem.

14. From the proposed Equation, which expresses the relation between x and y , find the relation of the fluxions $\dot{x}$ and $\dot{y}$, (by Prob. 1.) and putting $\dot{x} = 1$, there will be had the value of $\dot{y}$, to which z is equal. Then substituting z for $\dot{y}$, by the help of the last Equation find the relation of the Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, (by Prob. 1.) and again substituting 1 for $\dot{x}$, there will be had the value of $\dot{z}$. These

being found make $\frac{1 - yy}{z} = CP$, $z \times CP = PL$, and $CP \times \sqrt{1 - yy}$

$= CL$; and C will be a Point in the Curve, any part of which KC is equal to the right Line CG , which is the difference of the tangents, drawn perpendicularly to the Curve Dd from the points C and K .

15. Ex. Let $ax = yy$ be the Equation which expresses the relation between AP and PD; and (by Prob. 1.) it will be first $ax = 2yy$, or $a = 2yz$. Then $2yz + 2yz = 0$, or $\frac{-xz}{y} = z$. Thence it is $CP = \frac{1-yy}{z} = y - \frac{4y^3}{aa}$, $PL = z \times CP = \frac{1}{2}a - \frac{2yy}{a}$, and $CL = \frac{aa - 4yy}{2aa} \sqrt{4yy - aa}$. And from CP and PL taking away y and x , there remains $CD = -\frac{4y^3}{aa}$,

and $AL = \frac{1}{2}a - \frac{3yy}{a}$. Now I take away y and x , because when CP and PL have affirmative values, they fall on the side of the point P towards D and A, and they ought to be diminished, by taking away the affirmative quantities PD and AP. But when they have negative values, they will fall on the contrary side of the point P, and then they must be encreased, which is also done by taking away the affirmative quantities PD and AP.

16. Now to know the Length of the Curve, in which the point C is found, between any two of its points K and C; we must seek the length of the Tangent at the point K, and subtract it from CD. As if K were the point, at which the Tangent is terminated, when $C\Delta$ and Δg , or 1 and z , are made equal, which therefore is situate in the Absciss itself AP; write 1 for z in the Equation $a = 2yz$, whence $a = 2y$. Therefore for y write $\frac{1}{2}a$ in the value of CD, that is in $-\frac{4y^3}{aa}$, and it comes out $-\frac{1}{2}a$. And this is the length of the Tangent at the point K, or of DG; the difference between which and the foregoing indefinite value of CD, is $\frac{4y^3}{aa} - \frac{1}{2}a$, that is GC, to which the part of the Curve KC is equal.

17. Now that it may appear what Curve this is, from AL (having first changed its sign, that it may become affirmative,) take AK, which will be $\frac{1}{2}a$, and there will remain $KL = \frac{3yy}{a} - \frac{1}{2}a$, which call t , and in the value of the line CL, which call v , write $\frac{4at}{3}$ for $4yy - aa$, and there will arise $\frac{2t}{3a} \sqrt{\frac{4}{3}at} = v$; or $\frac{16t^3}{27a} = vv$, which is an Equation to a Parabola of the second kind, as was found before.

18. When the relation between t and v cannot conveniently be reduced to an Equation, it may be sufficient only to find the lengths PC and PL. As if for the relation between AP and PD the Equation $3a^2x + 3a^2y - y^3 = 0$ were assumed; from hence (by Prob. 1.) first there arises $a^2 + a^2z - y^2z = 0$, then $aa\dot{z} - 2yy\dot{z} - y^2\dot{z} = 0$, and therefore it is $z = \frac{aa}{yy - aa}$, and $\dot{z} = \frac{2yy\dot{y}}{aa - yy}$. Whence are given $PC = \frac{1 - yy}{z}$, and $PL = z \times PC$, by which the point C is determined, which is in the Curve. And the length of the Curve, between two such points, will be known by the difference of the two corresponding Tangents, DC or $PC - y$.

19. For Example, if we make $a = 1$, and in order to determine some point C of the Curve, we take $y = 2$; then AP or x becomes $\frac{y^3 - 3a^2y}{3aa} = \frac{8 - 6}{6} = \frac{1}{3}$, $z = \frac{1}{3}$, $\dot{z} = -\frac{4}{9}$, $PC = -2$, and $PL = -\frac{2}{3}$. Then to determine another point, if we take $y = 3$, it will be $AP = 6$, $z = \frac{1}{8}$, $\dot{z} = -\frac{3}{8}$, $PC = -84$, and $PL = -10\frac{1}{8}$. Which being had, if y be taken from PC, there will remain -4 in the first case, and -87 in the second, for the lengths DC; the difference of which 83 is the length of the Curve, between the two points found C and c.

20. These are to be thus understood, when the Curve is continued between the two points C and c, or between K and C, without that Term or Limit, which we call'd its Cuspid. For when one or more such terms come between those points, (which terms are found by the determination of the greatest or least PC or DC,) the lengths of each of the parts of the Curve, between them and the points C or K, must be separately found, and then added together.

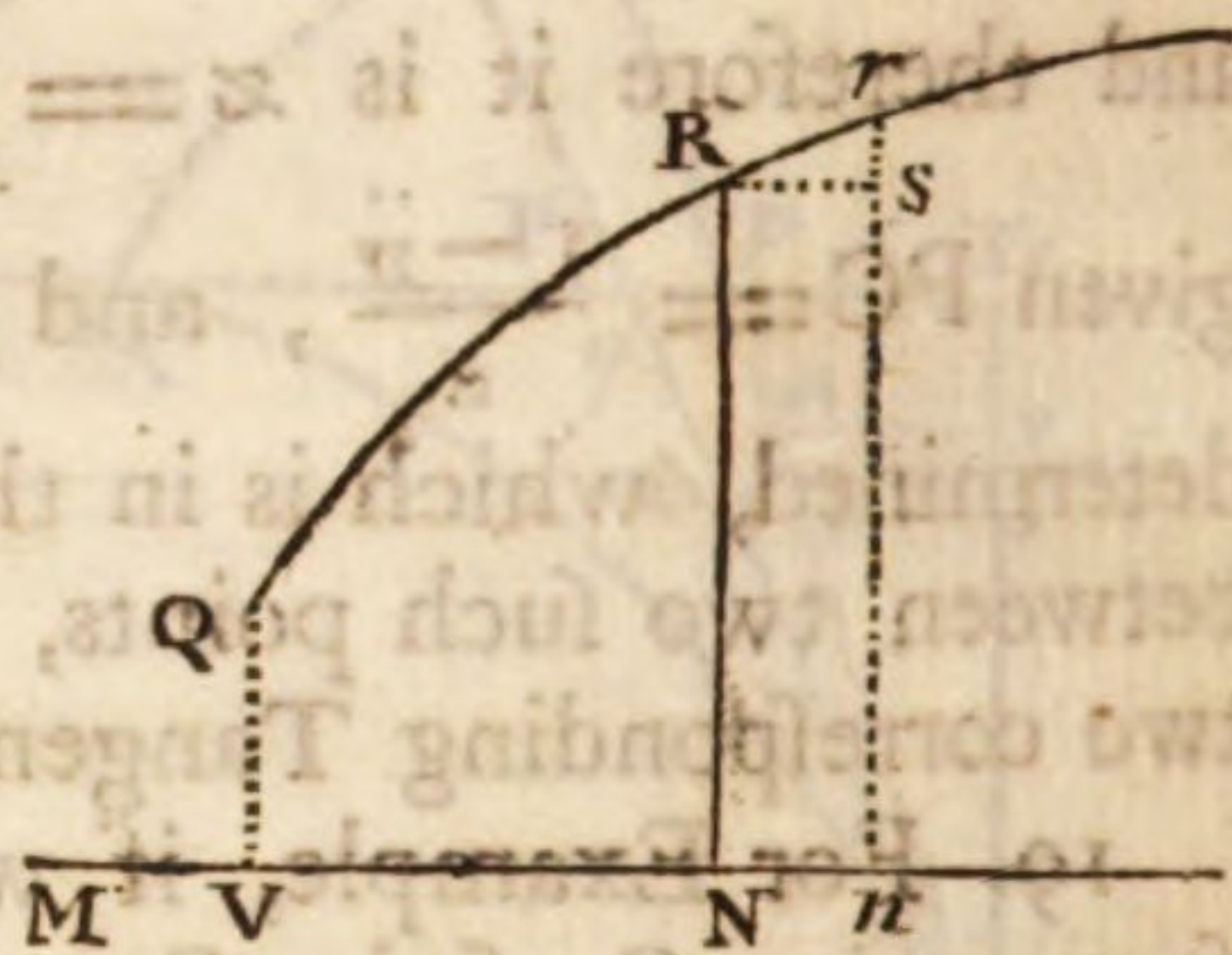
PROB. XI.

To find as many Curves as you please, whose Lengths may be compared with the Length of any Curve proposed, or with its Area applied to a given Line, by the help of finite Equations.

1. It is performed by involving the Length, or the Area of the proposed Curve, in the Equation which is assumed in the foregoing Problem, to determine the relation between AP and PD (Figure Art. 12. pag. 126.) But that z and $\dot{z}$ may be thence derived, (by

Prob. 1.) the fluxion of the Length, or of the Area, must be first discover'd.

2. The fluxion of the Length is determin'd by putting it equal to the square-root of the sum of the squares of the fluxion of the Absciss and of the Ordinate. For let RN be the perpendicular Ordinate, moving upon the Absciss MN, and let QR be the proposed Curve, at which RN is terminated. Then calling MN $= s$, NR $= t$, and QR $= v$, and their Fluxions $\dot{s}$, $\dot{t}$, and $\dot{v}$ respectively; conceive the Line NR to move into the place nr infinitely near the former, and letting fall Rs perpendicular to nr , then Rs, sr , and Rr will be the contemporaneous moments of the lines MN, NR, and QR, by the accession of which they become Mn, nr, and Qr. And as these are to each other as the fluxions of the same lines, and because of the right Angle Rsr, it will be $\sqrt{Rs^2 + sr^2} = Rr$, or $\sqrt{s^2 + t^2} = v$.



3. But to determine the fluxions $\dot{s}$ and $\dot{t}$ there are two Equations required; one of which is to define the relation between MN and NR, or s and t , from whence the relation between the fluxions $\dot{s}$ and $\dot{t}$ is to be derived; and another which may define the relation between MN or NR in the given Figure, and of AP or x in that required, from whence the relation of the fluxion $\dot{s}$ or $\dot{t}$ to the fluxion $\dot{x}$ or 1 may be discover'd.

4. Then $\dot{v}$ being found, the fluxions $\dot{y}$ and $\dot{z}$ are to be sought by a third assumed Equation, by which the length PD or y may be defined. Then we are to take $PC = \frac{1 - \dot{y}\dot{y}}{\dot{z}}$, $PL = \dot{y} \times PC$, and $DC = PC - y$, as in the foregoing Problem.

5. Ex. 1. Let $as - ss = tt$ be an Equation to the given Curve QR, which will be a Circle; $xx = as$ the relation between the lines AP and MN, and $\frac{2}{3}v = y$, the relation between the length of the Curve given QR, and the right Line PD. By the first it will be $as - 2ss = 2tt$, or $\frac{a - 2s}{2t} = \dot{t}$. And thence $\frac{as}{2t} = \sqrt{s^2 + t^2} = \dot{v}$. By the second it is $2x = as$, and therefore $\frac{x}{t} = \dot{v}$. And by the third $\frac{2}{3}\dot{v} = \dot{y}$, that is, $\frac{2x}{3t} = \dot{z}$, and hence $\frac{2}{3t} - \frac{2x\dot{t}}{3t^2} = \dot{z}$. Which being

being found, you must take $PC = \frac{1 - \ddot{y}}{z}$, $PL = \dot{y} \times PC$, and $DC = PC - y$, or $PC - \frac{1}{3}QR$. Where it appears, that the length of the given Curve QR cannot be found, but at the same time the length of the right Line DC must be known, and from thence the length of the Curve, in which the point C is found; and so on the contrary.

6. Ex. 2. The Equation $as - ss = tt$ remaining, make $x = s$, and $vv - 4ax = 4ay$. And by the first there will be found $\frac{as}{2t} = \dot{v}$, as above. But by the second $1 = \dot{s}$, and therefore $\frac{a}{2t} = \dot{v}$. And by the third $2vv - 4a = 4ay$, or (eliminating $\dot{v}$) $\frac{v}{4t} - 1 = z$.

Then from hence $\frac{\dot{v}}{4t} - \frac{\dot{v}t}{4tt} = \dot{z}$.
 7. Ex. 3. Let there be suppos'd three Equations, $aa = st$, $a + 3s = x$, and $x + v = y$. Then by the first, which denotes an Hyperbola, it is $0 = st + ts$, or $-\frac{st}{s} = \dot{t}$, and therefore $\frac{1}{3}\sqrt{ss + tt} = \dot{v}$. By the second it is $3\dot{s} = 1$, and therefore $\frac{1}{3}\sqrt{ss + tt} = \dot{v}$. And by the third it is $1 + v = y$, or $1 + \frac{1}{3}\sqrt{ss + tt} = z$; then it is from hence $\dot{w} = \dot{z}$, that is, putting $\dot{w}$ for the Fluxion of the radical $\frac{1}{3}\sqrt{ss + tt}$, which if it be made equal to w , or $\frac{1}{3} + \frac{tt}{9s} = ww$, there will arise from thence $\frac{2tt}{9s^2} = 2ww$. And first substituting $-\frac{st}{s}$ for t , then $\frac{1}{3}$ for s , and dividing by $2w$, there will arise $\frac{-2tt}{27ws^2} = \dot{w} = \dot{z}$. Now y and z being found, the rest is perform'd as in the first Example.

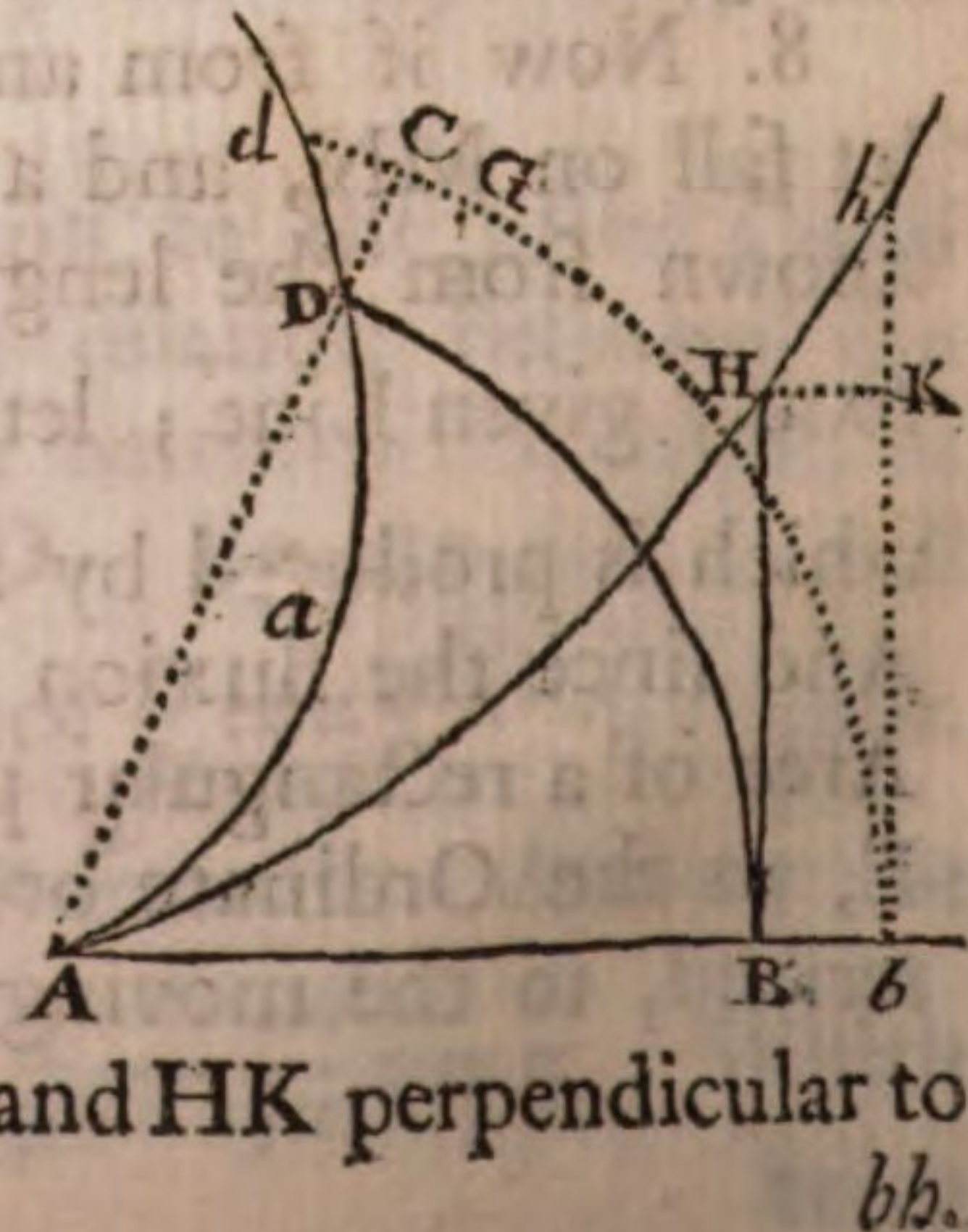
8. Now if from any point Q of a Curve, a perpendicular QV is let fall on MN , and a Curve is to be found whose length may be known from the length which arises by applying the Area $QRNV$ to any given Line; let that given Line be call'd E , the length $\frac{QRNV}{E}$ which is produced by such application be call'd v , and its fluxion $\dot{v}$. And since the fluxion of the Area $QRNV$ is to the Fluxion of the Area of a rectangular parallelogram made upon VN , with the height E , as the Ordinate or moving line $NR = t$, by which this is described, to the moving Line E , by which the other is described in the

the same time; and the fluxions $\dot{v}$ and $\dot{s}$ of the lines v and MN , (or s), or of the lengths which arise by applying those Areas to the given Line E , are in the same ratio; it will be $\dot{v} = \frac{\dot{s}t}{E}$. Therefore by this Rule the value of $\dot{v}$ is to be inquired, and the rest to be perform'd as in the Examples aforegoing.

9. Ex. 4. Let QR be an Hyperbola which is defined by this Equation, $aa + \frac{ass}{c} = tt$; and thence arises (by Prob. 1.) $\frac{ass}{c} = tt$, or $\frac{ass}{ct} = t$. Then if for the other two Equations are assumed $x = s$ and $y = v$; the first will give $t = s$, whence $\dot{v} = \frac{\dot{s}t}{E} = \frac{\dot{s}}{E}$; and the latter will give $\dot{y} = \dot{v}$, or $z = \frac{t}{E}$, then from hence $\dot{z} = \frac{\dot{t}}{E}$, and substituting $\frac{ass}{ct}$ or $\frac{as}{ct}$ for t , it becomes $\dot{z} = \frac{as}{Ect}$. Now y and z being found, make $\frac{1-yy}{z} = CP$, and $y \times CP = PL$, as before, and thence the Point C will be determin'd, and the Curve in which all such points are situated: The length of which Curve will be known from the length DC , which is equivalent to $CP - v$, as is sufficiently shewn before.

10. There is also another method, by which the Problem may be resolved; and that is by finding Curves whose fluxions are either equal to the fluxion of the proposed Curve, or are compounded of the fluxion of that, and of other Lines. And this may sometimes be of use, in converting mechanical Curves into equable Geometrical Curves; of which thing there is a remarkable Example in spiral lines.

11. Let AB be a right Line given in position, BD an Arch moving upon AB as an Absciss, and yet retaining A as its Center, ADd a Spiral, at which that arch is continually terminated, bd an arch indefinitely near it, or the place into which the arch BD by its motion next arrives, DC a perpendicular to the arch bd , dG the difference of the arches, AH another Curve equal to the Spiral AD , BH a right Line moving perpendicularly upon AB , and terminated at the Curve AH , bb the next place into which that right Line moves, and HK perpendicular to



bb. And in the infinitely little triangles *DCd* and *HKb*, since *DC* and *HK* are equal to the same third Line *Bb*, and therefore equal to each other, and *Dd* and *Hb* (by hypothesis) are correspondent parts of equal Curves, and therefore equal, as also the angles at *C* and *K* are right angles; the third sides *dC* and *bK* will be equal also. Moreover since it is $AB : BD :: Ab : bC :: Ab - AB (Bb) : bC - BD (CG)$; therefore $\frac{BD \times Bb}{AB} = CG$. If this be taken away from *dG*, there will remain $dG - \frac{BD \times Bb}{AB} = dC = bK$. Call therefore $AB = z$, $BD = v$, and $BH = y$, and their fluxions $\dot{z}$, $\dot{v}$, and $\dot{y}$ respectively, since *Bb*, *dG*, and *bK* are the contemporaneous moments of the same, by the accession of which they become *Ab*, *bd*, and *bb*, and therefore are to each other as the fluxions. Therefore for the moments in the last Equation let the fluxions be substituted, as also the letters for the Lines, and there will arise $\dot{v} - \frac{v\dot{z}}{z} = \dot{y}$. Now of these fluxions, if $\dot{z}$ be suppos'd equable, or the unit to which the rest are refer'd, the Equation will be $\dot{v} - \frac{v}{z} = \dot{y}$.

12. Wherefore the relation between *AB* and *BD*, (or between z and v), being given by any Equation, by which the Spiral is defined, the fluxion $\dot{v}$ will be given, (by Prob. 1.) and thence also the fluxion $\dot{y}$, by putting it equal to $\dot{v} - \frac{v}{z}$. And (by Prob. 2.) this will give the line y , or *BH*, of which it is the fluxion.

13. Ex. 1. If the Equation $\frac{zx}{a} = v$ were given, which is to the Spiral of *Archimedes*, thence (by Prob. 1.) $\frac{2zx}{a} = \dot{v}$. From hence take $\frac{v}{z}$, or $\frac{z}{a}$, and there will remain $\frac{z}{a} = \dot{y}$, and thence (by Prob. 2.) $\frac{zx}{2a} = y$. Which shews the Curve *AH*, to which the Spiral *AD* is equal; to be the Parabola of *Apollonius*, whose Latus rectum is $2a$; or whose Ordinate *BH* is always equal to half the Arch *BD*.

14. Ex. 2. If the Spiral be proposed which is defined by the Equation $z^3 = av^2$, or $v = \frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, there arises (by Prob. 1.) $\frac{3z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{v}$, from which if you take $\frac{v}{z}$, or $\frac{z^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, there will remain $\frac{z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$, and thence (by Prob. 2.) will be produced $\frac{z^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = y$. That is, $\frac{1}{3}BD = BH$, *AH* being a Parabola of the second kind.

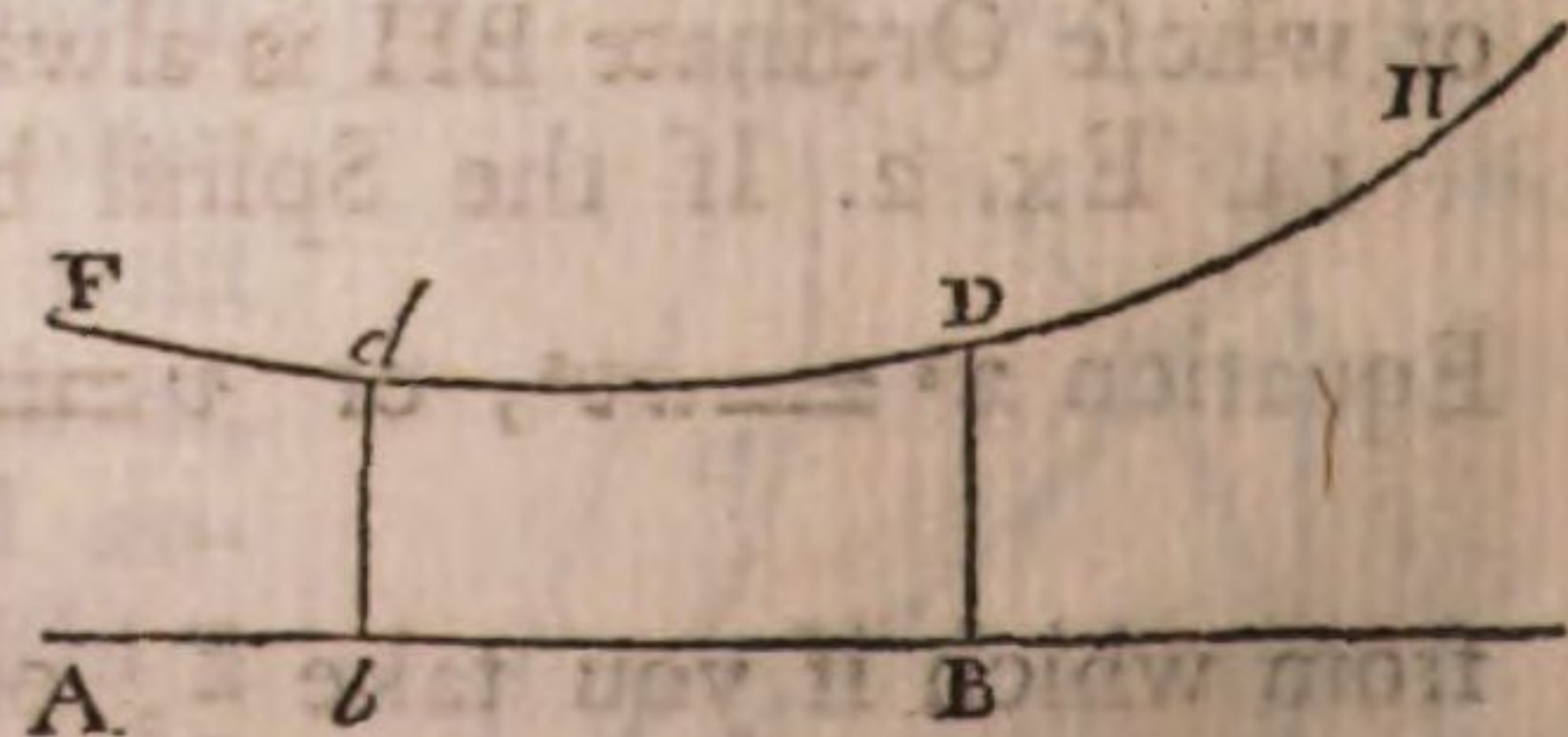
15. Ex. 3. If the Equation to the Spiral be $z\sqrt{\frac{a+z}{c}} = v$, thence (by Prob. 1.) $\frac{2a+3z}{2\sqrt{ac+cz}} = \dot{v}$; from whence if you take away $\frac{v}{z}$ or $\sqrt{\frac{a+z}{c}}$, there will remain $\frac{z}{2\sqrt{ac+cz}} = \dot{y}$. Now since the quantity generated by this fluxion $\dot{y}$ cannot be found by Prob. 2. unless it be resolved into an infinite Series; according to the tenor of the Scholium to Prob. 9. I reduce it to the form of the Equations in the first column of the Tables, by substituting z^η for z ; then it becomes $\frac{z^{2\eta-1}}{2\sqrt{ac+cz^\eta}} = \dot{y}$, which Equation belongs to the 2d Species of the 4th Order of Table 1. And by comparing the terms, it is $d = \frac{1}{2}$, $e = ac$, and $f = c$, so that $\frac{z-2a}{3c} \sqrt{ac+cz} = t = y$. Which Equation belongs to a Geometrical Curve AH, which is equal in length to the Spiral AD.

PROB. XII.

To determine the Lengths of Curves.

1. In the foregoing Problem we have shewn, that the Fluxion of a Curve-line is equal to the square-root of the sum of the squares of the Fluxions of the Absciss and of the perpendicular Ordinate. Wherefore if we take the Fluxion of the Absciss for an uniform and determinate measure, or for an Unit to which the other Fluxions are to be refer'd, and also if from the Equation which defines the Curve, we find the Fluxion of the Ordinate, we shall have the Fluxion of the Curve-line, from whence (by Problem 2.) its Length may be deduced.

2. Ex. 1. Let the Curve FDH be proposed, which is defined by the Equation $\frac{z^3}{aa} + \frac{aa}{12z} = y$; making the Absciss AB = z , and the moving Ordinate DB = y . Then from the Equation will be had, (by Prob. 1.) $\frac{3zz}{aa} - \frac{aa}{12zz} = \dot{y}$, the fluxion of z being 1, and $\dot{y}$ being the fluxion of y . Then adding the squares of the fluxions, the sum will be $\frac{9z^4}{a^4} + \frac{1}{4} + \frac{a^4}{144z^4} = \dot{t}^2$, and extracting the root, $\frac{3zz}{aa} + \frac{aa}{12zz} = \dot{t}$,



$\equiv t$, and thence (by Prob. 2.) $\frac{z^3}{aa} - \frac{aa}{12z} \equiv t$. Here t stands for the fluxion of the Curve, and t for its Length.

3. Therefore if the length dD of any portion of this Curve were required, from the points d and D let fall the perpendiculars db and DB to AB , and in the value of t substitute the quantities Ab and AB severally for z , and the difference of the results will be dD the Length required. As if $Ab \equiv \frac{1}{2}a$, and $AB \equiv a$, writing $\frac{1}{2}a$ for z , it becomes $t \equiv -\frac{a}{24}$; then writing a for z , it becomes $t \equiv \frac{11a}{12}$, from whence if the first value be taken away, there will remain $\frac{23a}{24}$ for the length dD . Or if only Ab be determin'd to be $\frac{1}{2}a$, and AB be look'd upon as indefinite, there will remain $\frac{z^3}{aa} - \frac{aa}{12z} + \frac{a}{24}$ for the value of dD .

4. If you would know the portion of the Curve which is represented by t , suppose the value of t to be equal to nothing, and there arises $z^4 \equiv \frac{a^4}{12}$, or $z \equiv \frac{a}{\sqrt[4]{12}}$. Therefore if you take $AB \equiv \frac{a}{\sqrt[4]{12}}$, and erect the perpendicular bd , the length of the Arch dD will be t , or $\frac{z^3}{aa} - \frac{aa}{12z}$. And the same is to be understood of all Curves in general.

5. After the same manner by which we have determin'd the length of this Curve, if the Equation $\frac{z^4}{a^3} + \frac{a^3}{32z^2} \equiv y$ be proposed, for defining the nature of another Curve; there will be deduced $\frac{z^4}{a^3} - \frac{a^3}{32z^2} \equiv t$; or if this Equation be proposed, $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} - \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} \equiv y$, there will arise $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} + \frac{1}{3}a^{\frac{1}{2}}z^{\frac{1}{2}} \equiv t$. Or in general, if it is $cz^{\theta} +$

$\frac{z^{\frac{1}{2}-\theta}}{4\theta c - 8\theta c} \equiv y$, where θ is used for representing any number, either Integer or Fraction, we shall have $cz^{\theta} - \frac{z^{\frac{1}{2}-\theta}}{4\theta c - 8\theta c} \equiv t$.

6. Ex. 2. Let the Curve be proposed which is defined by this Equation $\frac{2aa + 2xz}{3aa} \sqrt{aa + xz} \equiv y$; then (by Prob. 1.) will be had $y \equiv \frac{4a^4x + 8a^2xz^2 + 4z^3}{3a^4}$, or exterminating y , $y \equiv \frac{2z}{aa} \sqrt{aa + xz}$. To the square of which add 1, and the sum will be $1 + \frac{4xz}{aa} + \frac{4z^2}{a^2}$, and

and its Root $1 + \frac{2xz}{aa} = \dot{t}$. Hence (by Prob. 2.) will be obtain'd $z + \frac{2z^3}{3a^2} = t$.

7. Ex. 3. Let a Parabola of the second kind be proposed, whose Equation is $z^3 = ay^2$, or $\frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}} = y$, and thence by Prob. 1. is derived

$\frac{3z^{\frac{1}{2}}}{2a^{\frac{1}{2}}} = \dot{y}$. Therefore $\sqrt{1 + \frac{9z}{4a}} = \sqrt{1 + \dot{y}\dot{y}} = \dot{t}$. Now since the

length of the Curve generated by the Fluxion $\dot{t}$ cannot be found by Prob. 2. without a reduction to an infinite Series of simple Terms, I consult the Tables in Prob. 9. and according to the Scholium belonging to it, I have $t = \frac{8a + 18z}{27} \sqrt{1 + \frac{9z}{4a}}$. And thus you may find the lengths of these Parabolas $z^5 = ay^4$, $z^7 = ay^6$, $z^9 = ay^8$, &c.

8. Ex. 4. Let the Parabola be proposed, whose Equation is $z^4 = ay^3$, or $\frac{z^{\frac{4}{3}}}{a^{\frac{1}{3}}} = y$; and thence (by Prob. 1.) will arise $\frac{4z^{\frac{1}{3}}}{3a^{\frac{1}{3}}} = \dot{y}$.

Therefore $\sqrt{1 + \frac{16z^{\frac{2}{3}}}{9a^{\frac{2}{3}}}} = \sqrt{\dot{y}\dot{y} + 1} = \dot{t}$. This being found, I

consult the Tables according to the aforesaid Scholium, and by comparing with the 2d Theorem of the 5th Order of Table 2, I have

$z^{\frac{1}{3}} = x$, $\sqrt{1 + \frac{16xx}{9a^{\frac{2}{3}}}} = v$, and $\frac{3}{2}s = t$. Where x denotes the Ab-

sciss, y the Ordinate, and s the Area of the Hyperbola, and t the length which arises by applying the Area $\frac{3}{2}s$ to linear unity.

9. After the same manner the lengths of the Parabolas $z^6 = ay^5$, $z^8 = ay^7$, $z^{10} = ay^9$, &c. may also be reduced to the Area of the Hyperbola.

10. Ex. 5. Let the Cissoïd of the Ancients be proposed, whose Equation is $\frac{aa - 2az + zz}{\sqrt{az - zz}} = y$, and thence (by Prob. 1.) $\frac{-a - 2z}{2xz}$

$\sqrt{az - zz} = y$, and therefore $\frac{a}{2z} \sqrt{\frac{a+3z}{z}} = \sqrt{\dot{y}\dot{y} + 1} = \dot{t}$;

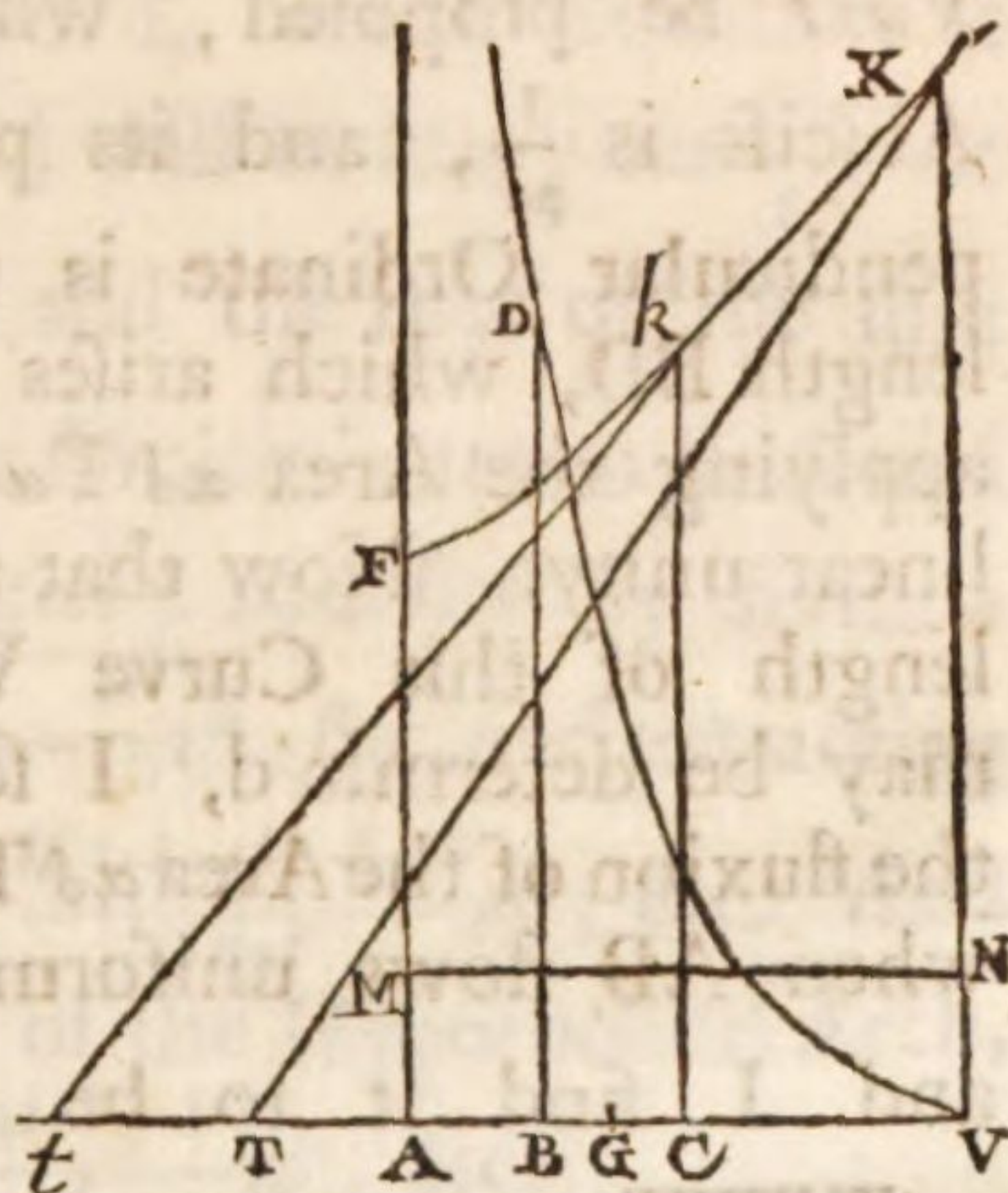
which by writing z^n for $\frac{1}{z}$ or z^{-1} , becomes $\frac{a}{2z} \sqrt{az^n + 3} = \dot{t}$, an Equation of the 1st Species of the 3d Order of Table 2; then comparing the Terms, it is $\frac{a}{2} = d$, $3 = e$, and $a = f$; so that

$z = \frac{1}{z^n} = x^2$, $\sqrt{a + 3xx} = v$, and $6s - \frac{2v^3}{x} = \frac{4de}{nf}$ into $\frac{v^3}{2ex} - s = t$.

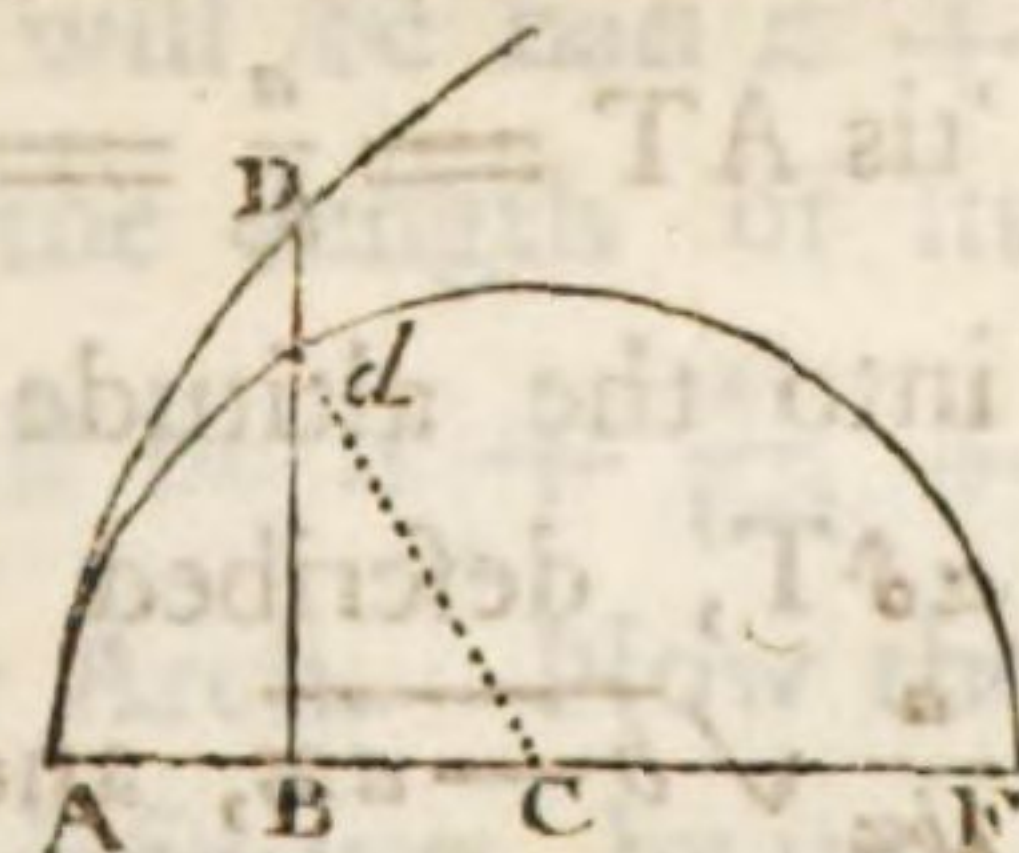
And

And taking a for Unity, by the Multiplication or Division of which, these Quantities may be reduced to a just number of Dimensions, it becomes $ax = xx$, $\sqrt{aa + 3xx} = v$, and $\frac{6s}{a} - \frac{27^3}{ax} = t$: Which are thus constructed.

11. The Cissoïd being VD , AV the Diameter of the Circle to which it is adapted, AF its Asymptote, and DB perpendicular to AV , cutting the Curve in D ; with the Semiaxis $AF = AV$, and the Semiparameter $AG = \frac{1}{3}AV$; let the Hyperbola FkK be described; and taking AC a mean Proportional between AB and AV , at C and V let Ck and VK drawn perpendicular to AV , cut the Hyperbola in k , and K , and let right Lines kt and KT touch it in those points, and cut AV in t and T ; and at AV let the Rectangle $AVNM$ be described, equal to the Space $TKkt$. Then the length of the Cissoïd VD will be sextuple of the Altitude VN .



12. Ex. 6. Supposing Ad to be an Ellipsis, which the Equation $\sqrt{ax - 2xz} = y$ represents; let the mechanical Curve AD be proposed of such a nature, that if Bd , or y , be produced till it meets this Curve at D , let BD be equal to the Elliptical Arch Ad . Now that the length of this may be determin'd, the Equation $\sqrt{ax - 2xz} = y$ will give



$\frac{a - 4z}{2\sqrt{ax - 2xz}} = y$, to the square of which if 1 be added, there arises $\frac{aa - 4az + 8zz}{4az - 8zz}$, the square of the fluxion of the arch Ad . To which if 1 be added again, there will arise $\frac{aa}{4az - 8zz}$, whose square-root $\frac{a}{2\sqrt{ax - 2xz}}$ is the fluxion of the Curve-line AD . Where if z be extracted out of the radical, and for z^{-1} be written z^n , there will be had $\frac{a}{2z^n\sqrt{az^n - 2}}$, a Fluxion of the 1st Species of the 4th Order of Table 2. Therefore the terms being collated, there will arise $d = \frac{1}{2}a$,

$e = -2$, and $f = a$; so that $z = \frac{1}{z^n} = x$, $\sqrt{ax - 2xx} = v$, and $\frac{8s}{a} - \frac{4xv}{a} + v = \frac{8de}{vff}$ into $s - \frac{1}{2}xv - \frac{fv}{4e} = t$.

13. The Construction of which is thus; that the right line dC being drawn to the center of the Ellipsis, a parallelogram may be made upon AC , equal to the sector ACd , and the double of its height will be the length of the Curve AD .

14. Ex. 7. Making $A\beta = \varphi$, (Fig. 1.) and $\alpha\delta$ being an Hyperbola, whose Equation is $\sqrt{-a + b\varphi\varphi} = \beta\delta$, and its tangent δT being drawn; let the Curve VdD be proposed, whose

Absciss is $\frac{1}{\varphi\varphi}$, and its perpendicular Ordinate is the length BD , which arises by applying the Area $\alpha\delta T\alpha$ to linear unity. Now that the length of this Curve VD may be determin'd, I seek the fluxion of the Area $\alpha\delta T\alpha$, when AB flows uniformly, and I find it to be $\frac{a}{4bz}$

$\sqrt{b - az}$, putting $AB = z$, and its fluxion unity. For

'tis $AT = \frac{a}{b\varphi} = \frac{a}{b} \sqrt{z}$, and its fluxion is $\frac{a}{2b\sqrt{z}}$, whose half drawn

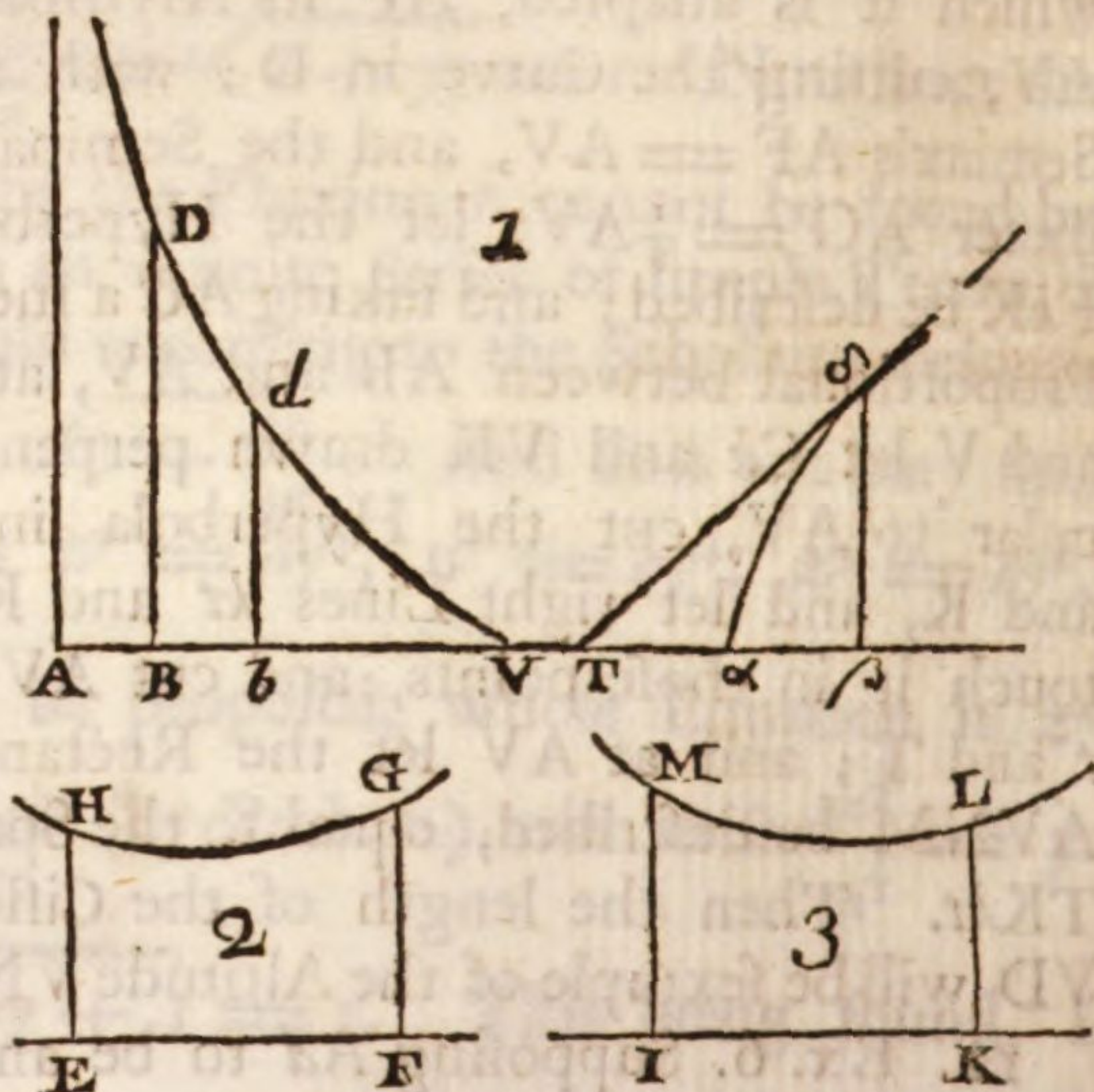
into the altitude $\beta\delta$, or $\sqrt{-a + \frac{b}{z}}$, is the fluxion of the Area $\alpha\delta T$, described by the Tangent δT . Therefore that fluxion is $\frac{a}{4bz} \sqrt{b - az}$, and this apply'd to unity becomes the fluxion of the

Ordinate BD . To the square of this $\frac{aab - a^3z}{16b^2z^2}$ add 1, the square of the fluxion BD , and there arises $\frac{aab - a^3z + 16b^2z^2}{16b^2z^2}$, whose root $\frac{1}{4bz}$

$\sqrt{a^2b - a^3z + 16b^2z^2}$ is the fluxion of the Curve VD . But this is a fluxion of the 1st Species of the 7th Order of Table 2: and the terms being collated, there will be $\frac{1}{4b} = \underline{d}$, $aab = \underline{e}$, $-a^3 = \underline{f}$,

$16b^2 = \underline{g}$, and therefore $z = \underline{x}$, and $\sqrt{a^2b - a^3x + 16b^2x^2} = \underline{v}$, (an Equation to one Conic Section, suppose HG , (Fig. 2.) whose Area $EFGH$ is s , where $EF = x$, and $FG = v$;) also $\frac{1}{z} = \underline{\xi}$,

and $\sqrt{16bb - a^3\xi + ab\xi^2} = \underline{\tau}$, (an Equation to another Conic Section,



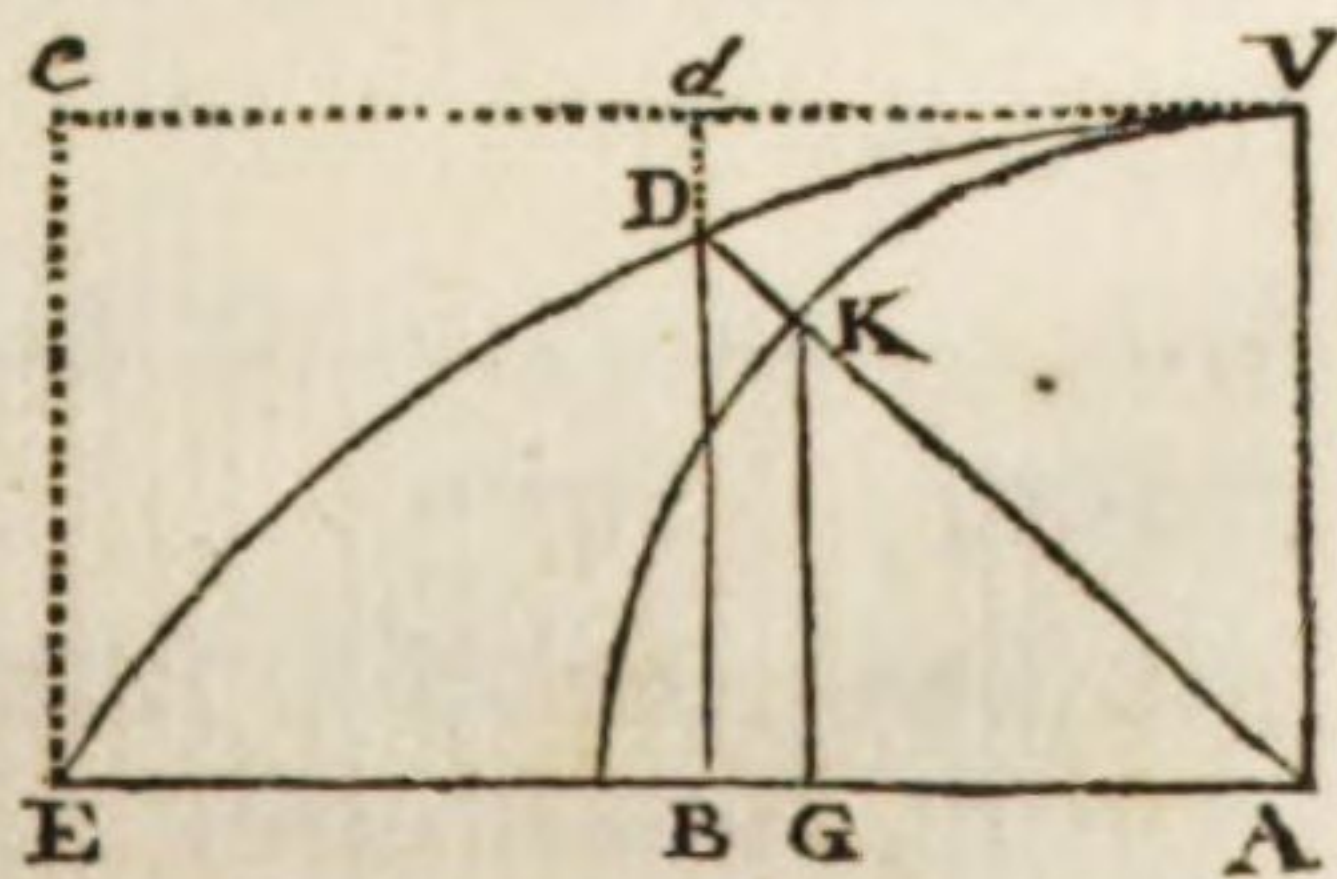
Section, suppose ML (Fig. 3.) whose Area IKLM is σ , where IK $= \xi$, and KL $= \tau$;) Lastly $\frac{2aabb\xi\tau - a^3b\tau - a^4v - 4aabb\sigma - 3zabbs}{64b^4 - a^4} = t$.

15. Wherefore that the length of any portion Dd of the Curve VD may be known, let fall db perpendicular to AB, and make Ab $= z$; and thence, by what is now found, seek the value of t . Then make AB $= z$, and thence also seek for t . And the difference of these two values of t will be the length Dd required.

16. Ex. 8. Let the Hyperbola be propos'd, whose Equation is $\sqrt{aa + bzz} = y$, and thence, (by Prob. 1.) will be had $\dot{y} = \frac{bz}{y}$, or $\frac{bz}{\sqrt{aa + bzz}}$. To the square of this add 1, and the root of the sum will be $\sqrt{\frac{aa + bzz + bbzz}{aa + bzz}} = t$. Now as this fluxion is not to be found in the Tables, I reduce it to an infinite Series; and first by division it becomes $t = \sqrt{1 + \frac{b^2}{a^2}z^2 - \frac{b^3}{a^4}z^4 + \frac{b^4}{a^6}z^6 - \frac{b^5}{a^8}z^8, \&c.}$ and extracting the root, $t = 1 + \frac{b^2}{2a^2}z^2 - \frac{4b^3 + b^4}{8a^4}z^4 + \frac{8b^4 + 4b^5 + b^6}{16a^6}z^6, \&c.}$ And hence (by Prob. 2.) may be had the length of the Hyperbolic Arch, or $t = z + \frac{b^2}{6a^2}z^3 - \frac{4b^3 + b^4}{40a^4}z^5 + \frac{8b^4 + 4b^5 + b^6}{112a^6}z^7, \&c.}$

17. If the Ellipsis $\sqrt{aa - bzz} = y$ were proposed, the Sign of b ought to be every where changed, and there will be had $z + \frac{b^2}{6a^2}z^3 + \frac{4b^3 - b^4}{40a^4}z^5 + \frac{8b^4 - 4b^5 + b^6}{112a^6}z^7, \&c.}$ for the length of its Arch. And likewise putting Unity for b , it will be $z + \frac{z^3}{6a^2} + \frac{3z^5}{40a^4} + \frac{5z^7}{112a^6}, \&c.}$ for the length of the Circular Arch. Now the numeral coefficients of this series may be found *ad infinitum*, by multiplying continually the terms of this Progression $\frac{1 \times 1}{2 \times 3}, \frac{3 \times 3}{4 \times 5}, \frac{5 \times 5}{6 \times 7}, \frac{7 \times 7}{8 \times 9}, \frac{9 \times 9}{10 \times 11}, \&c.}$

18. Ex. 9. Lastly, let the Quadratrix VDE be proposed, whose Vertex is V, A being the Center, and AV the semidiameter of the interior Circle, to which it is adapted, and the Angle VAE being a right Angle. Now any right Line AKD being drawn through A, cutting the Circle in K, and the Quadratrix in D, and the perpendiculars KG, DB being let fall to AE; call AV $= a$, AG $= z$, VK $= x$, and BD $= y$, and it



will be as in the foregoing Example, $x = z + \frac{z^3}{6a^2} + \frac{3z^5}{40a^4} + \frac{5z^7}{112a^6}$,
 &c. Extract the root z , and there will arise $z = x - \frac{x^3}{6a^2} + \frac{x^5}{120a^4}$
 $- \frac{x^7}{5040a^6}$, &c. whose Square subtract from AKq , or a^2 , and the
 root of the remainder $a - \frac{x^2}{2a} + \frac{x^4}{24a^3} - \frac{x^6}{720a^5}$, &c. will be GK .
 Now whereas by the nature of the Quadratrix 'tis $AB = VR = x$,
 and since it is $AG : GK :: AB : BD$ (y), divide $AB \times GK$ by AG ,
 and there will arise $y = a - \frac{xx}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5}$, &c. And thence,
 (by Prob. 1.) $y = -\frac{2x}{3a} - \frac{4x^3}{45a^3} - \frac{4x^5}{315a^5}$, &c. to the square of
 which add 1, and the root of the sum will be $1 + \frac{2xx}{9aa} + \frac{14x^4}{405a^4}$
 $+ \frac{604x^6}{127575a^6}$, &c. $= t$. Whence (by Prob. 2.) t may be obtain'd,
 or the Arch of the Quadratrix; viz. $VD = x + \frac{2x^3}{27a^2} + \frac{14x^5}{2825a^4} +$
 $\frac{604x^7}{893025a^6}$, &c.



THE

THE
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AND
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OR,
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THE Reader is desired to correct the following Errors, which I hope will be thought but few, and such as in works of this kind are hardly to be avoided. 'Tis here necessary to take notice of even literal Mistakes, which in subjects of this nature are often very material. That the Errors are so few, is owing to the kind assistance of a skilful Friend or two, who supply'd my frequent absence from the Press; as also to the care of a diligent Printer.

ERRATA.

In the Preface, pag. xiii. lin. 3. read which is here subjoin'd. Ibid. l. 5. for matter read manner. Pag. xxiii. l. ult. for Preface, &c. read Conclusion of this Work. P. 7. l. 31. for $\div$ read $=$. P. 15. l. 9. read $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. P. 17. l. 17. read $-\frac{2}{9x}$. P. 32. l. 21. read $\frac{y}{x}$. P. 35. l. 3. for $10xy^4$ read $10x^4y$. P. 62. l. 4. read $\frac{x+z^3}{z}$. P. 63. l. 31. for y read $\dot{y}$. Ibid. l. ult. for $-\frac{yz}{w}$ read $-\frac{y\dot{z}}{w}$. P. 64. l. 9. for 2 read z . Ibid. l. 30. read $\dot{z}$. P. 82. l. 17. read $2\dot{z}z$. P. 87. l. 22. read $+2a^{\frac{3}{2}}x^{\frac{1}{2}}$. Ibid. l. 22, 24. read AFDB. P. 88. l. 27. read $\dot{z}$. P. 92. l. 5. read $+\frac{2a^2}{5e^5}$. Ibid. l. 21. for z read $\dot{z}$. P. 104. l. 8. read $6ne^2$. P. 109. l. 33. dele as often. P. 110. l. 29. read $\sqrt{\frac{a^4}{a^2+z^2}} = \sqrt{AQ \times EC} = HA = AB = x$, and $\sqrt{a^2-x^2} =$. P. 113. l. 17. for Parabola read Hyperbola. P. 119. l. 12. read $CE \times \dot{ID} =$ to the Fluxion of the Area ACEG, and $ID \times \dot{IP} =$ to. P. 131. l. 8. read $+\frac{as}{2t}$. Ibid. l. 19. read $\frac{tt}{9ss}$. P. 135. l. 15. read Ab. P. 138. l. 9. read Absciss AB. P. 145. l. penult. read $7X^{-3}$. P. 149. l. 20. read which in. P. 157. l. 13. read $a \times$. P. 168. l. 5. read $\frac{1}{2}ax$. P. 171. l. 17. for B^3 read β^3 . P. 177. l. 15. read z^{m+2} . P. 189. l. 30. for the last P^{-3} read P^{-4} . P. 204. l. 16. read to zm . P. 213. l. 7. for Species read Series. P. 229. l. 21. for x^{-3} read x^{-4} . Ibid. l. 24. for x^{-4} read x^{-5} . P. 234. l. 2. for yy read y . P. 236. l. 26. read generating. P. 243. l. 29. read $-ax^2yy^{-1}$. P. 284. l. ult. read $\frac{y}{x}$. P. 289. l. 17. for right read left. P. 295. l. 1, 2. read $-\frac{1}{6}x^4 + \frac{5}{12}ax^4$. P. 297. l. 19. for yx^{-1} read yx^{-2} . P. 298. l. 14. read $-y$. P. 304. l. 20, 21. dele $+bc$. P. 309. l. 18. read a^{m-3} . P. 317. l. ult. read a^2j^2 .

The Description of the First Part, or, the Relation of the Planets
being given, to determine the Relation of the Planets.

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Part II. of particular Solutions: with a preparation to the
general Solution, by which it is distributed into three

The Reader is desired to consider the following Errors, which I hope will be thought
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THE
METHOD of FLUXIONS
AND
INFINITE SERIES.

ANNOTATIONS on the Introduction :

OR,

The Resolution of Equations by INFINITE SERIES.

SECT. I. *Of the Nature and Construction of Infinite
or Converging Series.*

I.  HE great Author of the foregoing Work begins it with a short Preface, in which he lays down his main design very concisely. He is not to be here understood, as if he would reproach the modern Geometricians with deserting the Ancients, or with abandoning their Synthetical Method of Demonstration, much less that he intended to disparage the Analytical Art; for on the contrary he has very much improved both Methods, and particularly in this Treatise he wholly applies himself to cultivate Analyticks, in which he has succeeded to universal applause and admiration. Not but that we shall find here some examples of the Synthetical Method likewise, which are very masterly and elegant. Almost all that remains of the ancient Geometry is indeed Synthetical, and proceeds by way of demonstrating truths already known, by shewing their dependence upon the Axioms, and other

other first Principles, either mediately or immediately. But the business of Analyticks is to investigate such Mathematical Truths as really are, or may be suppos'd at least to be unknown. It assumes those Truths as granted, and argues from them in a general manner, till after a series of argumentation, in which the several steps have a necessary connexion with each other, it arrives at the knowledge of the proposition required, by comparing it with something really known or given. This therefore being the Art of Invention, it certainly deserves to be cultivated with the utmost industry. Many of our modern Geometricians have been persuaded, by considering the intricate and labour'd Demonstrations of the Ancients, that they were Masters of an Analysis purely Geometrical, which they studiously conceal'd, and by the help of which they deduced, in a direct and scientific manner, those abstruse Propositions we so much admire in some of their writings, and which they afterwards demonstrated Synthetically. But however this may be, the loss of that Analysis, if any such there were, is amply compensated, I think, by our present Arithmetical or Algebraical Analysis, especially as it is now improved, I might say perfected, by our sagacious Author in the Method before us. It is not only render'd vastly more universal and extensive than that other in all probability could ever be, but is likewise a most compendious Analysis for the more abstruse Geometrical Speculations, and for deriving Constructions and Synthetical Demonstrations from thence; as may abundantly appear from the ensuing Treatise.

2. The conformity or correspondence, which our Author takes notice of here, between his new-invented Doctrine of infinite Series, and the commonly received Decimal Arithmetick, is a matter of considerable importance, and well deserves, I think, to be set in a fuller Light, for the mutual illustration of both; which therefore I shall here attempt to perform. For Novices in this Doctrine, tho' they may already be well acquainted with the Vulgar Arithmetick, and with the Rudiments of the common Algebra, yet are apt to apprehend something abstruse and difficult in infinite Series; whereas indeed they have the same general foundation as Decimal Arithmetick, especially Decimal Fractions, and the same Notion or Notation is only carry'd still farther, and render'd more universal. But to shew this in some kind of order, I must inquire into these following particulars. First I must shew what is the true Nature, and what are the genuine Principles, of our common Scale of Decimal Arithmetick. Secondly what is the nature of other particular Scales, which have been, or
may

may be, occasionally introduced. Thirdly, what is the nature of a general Scale, which lays the foundation for the Doctrine of infinite Series. Lastly, I shall add a word or two concerning that Scale of Arithmetick in which the Root is unknown, and therefore proposed to be found; which gives occasion to the Doctrine of Affected Equations.

First then as to the common Scale of Decimal Arithmetick, it is that ingenious Artifice of expressing, in a regular manner, all conceivable Numbers, whether Integers or Fractions, Rational or Surd, by the several Powers of the number *Ten*, and their Reciprocals; with the assistance of other small Integer Numbers, not exceeding Nine, which are the Coefficients of those Powers. So that Ten is here the Root of the Scale, which if we denote by the Character *X*, as in the *Roman* Notation, and its several Powers by the help of this Root and Numeral Indexes, ($X^1 = 10$, $X^2 = 100$, $X^3 = 1000$, $X^4 = 10000$, &c.) as is usual; then by assuming the Coefficients 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, as occasion shall require, we may form or express any Number in this Scale. Thus for instance $5X^4 + 7X^3 + 4X^2 + 8X^1 + 3X^0$ will be a particular Number express'd by this Scale, and is the same as 57483 in the common way of Notation. Where we may observe, that this last differs from the other way of Notation only in this, that here the several Powers of *X* (or *Ten*) are suppress'd, together with the Sign of Addition +, and are left to be supply'd by the Understanding. For as those Powers ascend regularly from the place of Units, (in which is always X^0 , or 1, multiply'd by its Coefficient, which here is 3,) the several Powers will easily be understood, and may therefore be omitted, and the Coefficients only need to be set down in their proper order. Thus the Number 7906538 will stand for $7X^6 + 9X^5 + 0X^4 + 6X^3 + 5X^2 + 3X^1 + 8X^0$, when you supply all that is understood. And the Number 1736 (by suppressing what may be easily understood,) will be equivalent to $X^3 + 7X^2 + 3X + 6$; and the like of all other Integer Numbers whatever, express'd by this Scale, or with this Root *X*, or *Ten*.

The same Artifice is uniformly carry'd on, for the expressing of all Decimal Fractions, by means of the Reciprocals of the several Powers of *Ten*, such as $\frac{1}{X} = 0,1$; $\frac{1}{X^2} = 0,01$; $\frac{1}{X^3} = 0,001$; &c. which Reciprocals may be intimated by negative Indices. Thus the Decimal Fraction 0,3172 stands for $3X^{-1} + 1X^{-2} + 7X^{-3} + 2X^{-4}$; and the mixt Number 526,384 (by supplying what is understood) becomes

becomes $5X^4 + 2X^3 + 6X^2 + 3X^{-1} + 8X^{-2} + 4X^{-3}$; and the infinite or interminate Decimal Fraction $0,9999999$, &c. stands for $9X^{-1} + 9X^{-2} + 9X^{-3} + 9X^{-4} + 9X^{-5} + 9X^{-6}$, &c. which infinite Series is equivalent to Unity. So that by this Decimal Scale, (or by the several Powers of Ten and their Reciprocals, together with their Coefficients, which are all the whole Numbers below Ten,) all conceivable Numbers may be express'd, whether they are integer or fracted, rational or irrational; at least by admitting of a continual progress or approximation *ad infinitum*.

And the like may be done by any other Scale, as well as the Decimal Scale, or by admitting any other Number, besides Ten, to be the Root of our Arithmetick. For the Root Ten was an arbitrary Number, and was at first assumed by chance, without any previous consideration of the nature of the thing. Other Numbers perhaps may be assign'd, which would have been more convenient, and which have a better claim for being the Root of the Vulgar Scale of Arithmetick. But however this may prevail in common affairs, Mathematicians make frequent use of other Scales; and therefore in the second place I shall mention some other particular Scales, which have been occasionally introduced into Computations.

The most remarkable of these is the Sexagenary or Sexagesimal Scale of Arithmetick, of frequent use among Astronomers, which expresses all possible Numbers, Integers or Fractions, Rational or Surd, by the Powers of *Sixty*, and certain numeral Coefficients not exceeding fifty-nine. These Coefficients, for want of peculiar Characters to represent them, must be express'd in the ordinary Decimal Scale. Thus if ξ stands for 60, as in the *Greek* Notation, then one of these Numbers will be $53\xi^2 + 9\xi^1 + 34\xi^0$, or in the Sexagenary Scale $53''$, $9'$, 34^0 , which is equivalent to 191374^0 in the Decimal Scale. Again, the Sexagesimal Fraction 53^0 , $9'$, $34''$, will be the same as $53\xi^{-1} + 9\xi^{-2} + 34\xi^{-3}$, which in Decimal Numbers will be $53,159444$, &c. *ad infinitum*. Whence it appears by the way, that some Numbers may be express'd by a finite number of Terms in one Scale, which in another cannot be express'd but by approximation, or by a progression of Terms *in infinitum*.

Another particular Scale that has been consider'd, and in some measure has been admitted into practice, is the Duodecimal Scale, which expresses all Numbers by the Powers of *Twelve*. So in common affairs we say a Dozen, a Dozen of Dozens or a Gross, a Dozen of Grosses or a great Gross, &c. And this perhaps would have been the most convenient Root of all others, by the Powers of which

to construct the popular Scale of Arithmetick ; as not being so big but that its Multiples, and all below it, might be easily committed to memory ; and it admits of a greater variety of Divisors than any Number not much greater than itself. Besides, it is not so small, but that Numbers express'd hereby would sufficiently converge, or by a few figures would arrive near enough to the Number required ; the contrary of which is an inconvenience, that must necessarily attend the taking too small a Number for the Root. And to admit this Scale into practice, only two single Characters would be wanting, to denote the Coefficients Ten and Eleven.

Some have consider'd the Binary Arithmetick, or that Scale in which *Two* is the Root, and have pretended to make Computations by it, and to find considerable advantages in it. But this can never be a convenient Scale to manage and express large Numbers by, because the Root, and consequently its Powers, are so very small, that they make no dispatch in Computations, or converge exceeding slowly. The only Coefficients that are here necessary are 0 and 1. Thus $1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$ is one of these Numbers, (or compendiously 110110,) which in the common Notation is no more than 54. Mr. *Leibnitz* imagin'd he had found great Mysteries in this Scale. See the Memoirs of the Royal Academy of *Paris*, Anno 1703.

In common affairs we have frequent recourse, though tacitly, to Millenary Arithmetick, and other Scales, whose Roots are certain Powers of Ten. As when a large Number, for the convenience of reading, is distinguish'd into Periods of three figures : As 382,735,628,490. Here 382, and 735, &c. may be consider'd as Coefficients, and the Root of the Scale is 1000. So when we reckon by Millions, Billions, Trillions, &c. a Million may be conceived as the Root of our Arithmetick. Also when we divide a Number into pairs of figures, for the Extraction of the Square-root ; into ternaries of figures for the Extraction of the Cube-root ; &c. we take new Scales in effect, whose Roots are 100, 1000, &c.

Any Number whatever, whether Integer or Fraction, may be made the Root of a particular Scale, and all conceivable Numbers may be express'd or computed by that Scale, admitting only of integral and affirmative Coefficients, whose number (including the Cypher 0) need not be greater than the Root. Thus in Quinary Arithmetick, in which the Scale is compos'd of the Powers of the Root 5, the Coefficients need be only the five Numbers 0, 1, 2, 3, 4, and yet all Numbers whatever are expressible by this Scale, at least by approximation,

mation, to what accuracy we please. Thus the common Number 2827,92 in this Arithmetick would be $4 \times 5^4 + 2 \times 5^3 + 3 \times 5^2 + 0 \times 5^1 + 2 \times 5^0 + 4 \times 5^{-1} + 3 \times 5^{-2}$; or if we may supply the several Powers of 5 by the Imagination only, as we do those of Ten in the common Scale, this Number will be 42302,43 in Quinary Arithmetick.

All vulgar Fractions and mixt Numbers are, in some measure, the expressing of Numbers by a particular Scale, or making the Denominator of the Fraction to be the Root of a new Scale. Thus $\frac{2}{3}$ is in effect $0 \times 3^0 + 2 \times 3^{-1}$; and $8\frac{3}{5}$ is the same as $8 \times 5^0 + 3 \times 5^{-1}$; and $25\frac{4}{9}$ reduced to this Notation will be $25 \times 9^0 + 4 \times 9^{-1}$, or rather $2 \times 9^1 + 7 \times 9^0 + 4 \times 9^{-1}$. And so of all other Fractions and mixt Numbers.

A Number computed by any one of these Scales is easily reduced to any other Scale assign'd, by substituting instead of the Root in one Scale, what is equivalent to it express'd by the Root of the other Scale. Thus to reduce Sexagenary Numbers to Decimals, because $60 = 6 \times 10$, or $\xi = 6X$, and therefore $\xi^2 = 36X^2$, $\xi^3 = 216X^3$, &c. by the substitution of these you will easily find the equivalent Decimal Number. And the like in all other Scales.

The Coefficients in these Scales are not necessarily confin'd to be affirmative integer Numbers less than the Root, (tho' they should be such if we would have the Scale to be regular,) but as occasion may require they may be any Numbers whatever, affirmative or negative, integers or fractions. And indeed they generally come out promiscuously in the Solution of Problems. Nor is it necessary that the Indices of the Powers should be always integral Numbers, but may be any regular Arithmetical Progression whatever, and the Powers themselves either rational or irrational. And thus (thirdly) we are come by degrees to the Notion of what is call'd an universal Series, or an indefinite or infinite Series. For supposing the Root of the Scale to be indefinite, or a general Number, which may therefore be represented by x , or y , &c. and assuming the general Coefficients a, b, c, d , &c. which are Integers or Fractions, affirmative or negative, as it may happen; we may form such a Series as this, $ax^4 + bx^3 + cx^2 + dx^1 + ex^0$, which will represent some certain Number, express'd by the Scale whose Root is x . If such a Number proceeds *in infinitum*, then it is truly and properly call'd an Infinite Series, or a Converging Series, x being then suppos'd greater than Unity. Such for example is $x + \frac{1}{2}x^{-1} + \frac{1}{3}x^{-2} + \frac{1}{4}x^{-3}$, &c. where the rest of the Terms are understood *ad infinitum*, and are insinuated by,

by, &c. And it may have any descending Arithmetical Progression for its Indices, as $x^m - \frac{2}{3}x^{m-1} + \frac{3}{4}x^{m-2} - \frac{4}{5}x^{m-3}$, &c.

And thus we have been led by proper gradations, (that is, by arguing from what is well known and commonly received, to what before appear'd to be difficult and obscure,) to the knowledge of infinite Series, of which the Learner will find frequent Examples in the sequel of this Treatise. And from hence it will be easy to make the following general Inferences, and others of a like nature, which will be of good use in the farther knowledge and practice of these Series; *viz.* That the first Term of every regular Series is always the most considerable, or that which approaches nearer to the Number intended, (denoted by the Aggregate of the Series,) than any other single Term: That the second is next in value, and so on: That therefore the Terms of the Series ought always to be disposed in this regular descending order, as is often inculcated by our Author: That when there is a Progression of such Terms *in infinitum*, a few of the first Terms, or those at the beginning of the Series, are or should be a sufficient Approximation to the whole; and that these may come as near to the truth as you please, by taking in still more Terms: That the same Number in which one Scale may be express'd by a finite number of Terms, in another cannot be express'd but by an infinite Series, or by approximation only, and *vice versâ*: That the bigger the Root of the Scale is, by so much the faster, *cæteris paribus*, the Series will converge; for then the Reciprocals of the Powers will be so much the less, and therefore may the more safely be neglected: That if a Series converges by increasing Powers, such as $ax + bx^2 + cx^3 + dx^4$, &c. the Root x of the Scale must be understood to be a proper Fraction, the lesser the better. Yet whenever a Series can be made to converge by the Reciprocals of Ten, or its Compounds, it will be more convenient than a Series that converges faster; because it will more easily acquire the form of the Decimal Scale, to which, in particular Cases, all Series are to be ultimately reduced. Lastly, from such general Series as these, which are commonly the result in the higher Problems, we must pass (by substitution) to particular Scales or Series, and those are finally to be reduced to the Decimal Scale. And the Art of finding such general Series, and then their Reduction to particular Scales, and last of all to the common Scale of Decimal Numbers, is almost the whole of the abstruser parts of Analyticks, as may be seen in a good measure by the present Treatise.

I took notice in the fourth place, that this Doctrine of Scales, and Series, gives us an easy notion of the nature of affected Equations, or shews us how they stand related to such Scales of Numbers. In the other Instances of particular Scales, and even of general ones, the Root of the Scale, the Coefficients, and the Indices, are all suppos'd to be given, or known, in order to find the Aggregate of the Series, which is here the thing required. But in affected Equations, on the contrary, the Aggregate and the rest are known, and the Root of the Scale, by which the Number is computed, is unknown and required. Thus in the affected Equation $5x^4 + 3x^3 + 0x^2 + 7x = 53070$, the Aggregate of the Series is given, *viz.* the Number 53070, to find x the Root of the Scale. This is easily discern'd to be 10, or to be a Number express'd by the common Decimal Scale, especially if we supply the several Powers of 10, where they are understood in the Aggregate, thus $5X^4 + 3X^3 + 0X^2 + 7X^1 + 0X^0 = 53070$. Whence by comparison 'tis $x = X = 10$. But this will not be so easily perceived in other instances. As if I had the Equation $4x^4 + 2x^3 + 3x^2 + 0x^1 + 2x^0 + 4x^{-1} + 3x^{-2} = 2827,92$ I should not so easily perceive that the Root x was 5, or that this is a Number express'd by Quinary Arithmetick, except I could reduce it to this form, $4 \times 5^4 + 2 \times 5^3 + 3 \times 5^2 + 0 \times 5^1 + 2 \times 5^0 + 4 \times 5^{-1} + 3 \times 5^{-2} = 2827,92$, when by comparison it would presently appear, that the Root sought must be 5. So that finding the Root of an affected Equation is nothing else, but finding what Scale in Arithmetick that Number is computed by, whose Result or Aggregate is given in the common Scale; which is a Problem of great use and extent in all parts of the Mathematicks. How this is to be done, either in Numeral, Algebraical, or Fluxional Equations, our Author will instruct us in its due place.

Before I dismiss this copious and useful Subject of Arithmetical Scales, I shall here make this farther Observation; that as all conceivable Numbers whatever may be express'd by any one of these Scales, or by help of an Aggregate or Series of Powers derived from any Root; so likewise any Number whatever may be express'd by some single Power of the same Root, by assuming a proper Index, integer or fracted, affirmative or negative, as occasion shall require. Thus in the Decimal Scale, the Root of which is 10, or X , not only the Numbers 1, 10, 100, 1000, &c. or 1, 0.1, 0.01, 0.001, &c. that is, the several integral Powers of 10 and their Reciprocals, may be express'd by the single Powers of X or 10, *viz.* X^0 , X^1 , X^2 , X^3 , &c. or X^0 , X^{-1} , X^{-2} , X^{-3} , &c. respectively, but also all the intermediate

mediate Numbers, as 2, 3, 4, &c. 11, 12, 13, &c. may be express'd by such single Powers of X or 10, if we assume proper Indices. Thus $2 = X_{0,30103, \&c.}$, $3 = X_{0,47712, \&c.}$, $4 = X_{0,60206, \&c.}$ &c. or $11 = X_{1,04139, \&c.}$, $12 = X_{1,07918, \&c.}$, $456 = X_{2,65896, \&c.}$ And the like of all other Numbers. These Indices are usually call'd the Logarithms of the Numbers (or Powers) to which they belong, and are so many Ordinal Numbers, declaring what Power (in order or succession) any given Number is, of any Root assign'd: And different Scales of Logarithms will be form'd, by assuming different Roots of those Scales. But how these Indices, Logarithms, or Ordinal Numbers may be conveniently found, our Author will likewise inform us hereafter. All that I intended here was to give a general Notion of them, and to shew their dependance on, and connexion with, the several Arithmetical Scales before described.

It is easy to observe from the *Arenarius* of *Archimedes*, that he had fully consider'd and discuss'd this Subject of Arithmetical Scales, in a particular Treatise which he there quotes, by the name of his *ἀρχαί*, or Principles; in which (as it there appears) he had laid the foundation of an Arithmetick of a like nature, and of as large an extent, as any of the Scales now in use, even the most universal. It appears likewise, that he had acquired a very general notion of the Doctrine and Use of Indices also. But how far he had accommodated an Algorithm, or Method of Operation, to those his Principles, must remain uncertain till that Book can be recover'd, which is a thing more to be wish'd than expected. However it may be fairly concluded from his great Genius and Capacity, that since he thought fit to treat on this Subject, the progress he had made in it was very considerable.

But before we proceed to explain our Author's methods of Operation with infinite Series, it may be expedient to enlarge a little farther upon their nature and formation, and to make some general Reflexions on their Convergency, and other circumstances. Now their formation will be best explain'd by continual Multiplication after the following manner.

Let the quantity $a + bx + cx^2 + dx^3 + ex^4, \&c.$ be assumed as a Multiplier, consisting either of a finite or an infinite number of Terms; and let also $\frac{p}{q} + x = 0$ be such a Multiplier, as will give the Root $x = -\frac{p}{q}$. If these two are multiply'd together, they will produce $\frac{ap}{q} + \frac{bp+aq}{q}x + \frac{cp+bq}{q}x^2 + \frac{dp+cq}{q}x^3 + \frac{ep+dq}{q}x^4, \&c. = 0;$

$= 0$; and if instead of x we here substitute its value $-\frac{p}{q}$, the Series will become $\frac{ap}{q} - \frac{bp+aq}{q} \times \frac{p}{q} + \frac{cp+bq}{q} \times \frac{p^2}{q^2} - \frac{dp+cq}{q} \times \frac{p^3}{q^3} + \frac{ep+dq}{q} \times \frac{p^4}{q^4}$, &c. $= 0$; or if we divide by $\frac{p}{q}$, and transpose, it will be $\frac{bp+aq}{q} - \frac{cp+bq}{q} \times \frac{p}{q} + \frac{dp+cq}{q} \times \frac{p^2}{q^2} - \frac{ep+dq}{q} \times \frac{p^3}{q^3}$, &c. $= a$: which Series, thus derived, may give us a good insight into the nature of infinite Series in general. For it is plain that this Series, (even though it were continued to infinity,) must always be equal to a , whatever may be supposed to be the values of p, q, a, b, c, d , &c. For $\frac{bp}{q}$, the first part of the first Term, will always be removed or destroy'd by its equal with a contrary Sign, in the second part of the second Term. And $\frac{cp}{q} \times \frac{p}{q}$, the first part of the second Term, will be removed by its equal with a contrary Sign, in the second part of the third Term, and so on: So as finally to leave $\frac{aq}{q}$, or a , for the Aggregate of the whole Series. And here it is likewise to be observ'd, that we may stop whenever we please, and yet the Equation will be good, provided we take in the *Supplement*, or a due part of the next Term. And this will always obtain, whatever the nature of the Series may be, or whether it be converging or diverging. If the Series be diverging, or if the Terms continually increase in value, then there is a necessity of taking in that Supplement, to preserve the integrity of the Equation. But if the Series be converging, or if the Terms continually decrease in any compound Ratio, and therefore finally vanish or approach to nothing; the Supplement may be safely neglected, as vanishing also, and any number of Terms may be taken, the more the better, as an Approximation to the Quantity a . And thus from a due consideration of this fictitious Series, the nature of all converging or diverging Series may easily be apprehended. Diverging Series indeed, unless when the afore-mention'd increasing Supplement can be assign'd and taken in, will be of no service. And this Supplement, in Series that commonly occur, will be generally so entangled and complicated with the Coefficients of the Terms of the Series, that altho' it is always to be understood, nevertheless, it is often impossible to be extricated and assign'd. But however, converging Series will always be of excellent use, as affording a convenient Approximation to the quantity required, when it cannot be otherwise exhibited. In these the Supplement aforesaid, tho'

tho' generally inextricable and unassignable, yet continually decreases along with the Terms of the Series, and finally becomes less than any assignable Quantity.

The same Quantity may often be exhibited or express'd by several converging Series; but that Series is to be most esteem'd that has the greatest Rate of Convergency. The foregoing Series will converge so much the faster, *cæteris paribus*, as p is less than q , or as the Fraction $\frac{p}{q}$ is less than Unity. For if it be equal to, or greater than Unity, it may become a diverging Series, and will diverge so much the faster, as p is greater than q . The Coefficients will contribute little or nothing to this Convergency or Divergency, if they are suppos'd to increase or decrease (as is generally the case) rather in a simple and Arithmetical, than a compound and Geometrical Proportion. To make some Estimate of the Rate of Convergency in this Series, and by analogy in any other of this kind, let k and l represent two Terms indefinitely, which immediately succeed each other in the progression of the Coefficients of the Multiplier $a + bx + cx^2 + dx^3$, &c. and let the number n represent the order or place of k . Then any Term of the Series indefinitely may be represented by $\pm \frac{lp + kq}{q^n} p^{n-1}$; where the Sign must be $+$ or $-$, according as n is an odd or an even Number. Thus if $n = 1$, then $k = a$, $l = b$, and the first Term will be $+\frac{bp + aq}{q}$. If $n = 2$, then $k = b$, $l = c$, and the second Term will be $-\frac{cp + bq}{q^2} p$. And so of the rest. Also if m be the next Term in the aforesaid progression after l , then $\pm \frac{lp + kq}{q^n} p^{n-1} \pm \frac{mp + lq}{q^{n+1}} p^n$ will be any two successive Terms in the same Series. Now in order to a due Convergency, the former Term absolutely consider'd, that is setting aside the Signs, should be as much greater than the succeeding Term, as conveniently may be. Let us suppose therefore that $\frac{lp + kq}{q^n} p^{n-1}$ is greater than $\frac{mp + lq}{q^{n+1}} p^n$, or (dividing all by the common factor $\frac{p^n}{q^n}$,) that $\frac{lp + kq}{p}$ is greater than $\frac{mp + lq}{q}$, or (multiplying both by pq ,) that $lpq + kq^2$ is greater than $mp^2 + lpq$, or (taking away the common lpq ,) that kq^2 is greater than mp^2 , or (by a farther Division,) that $\frac{k}{m} \times \frac{q^2}{p^2}$ is greater than unity; and as much greater as may be.

This will take effect on a double account; first, the greater k is in respect of m , and secondly, the greater q^2 is in respect of p^2 . Now in the Multiplier $a + bx + cx^2 + dx^3$, &c. if the Coefficients a, b, c , &c. are in any decreasing Progression, then k will be greater than l , which is greater than m ; so that *a fortiori* k will be greater than m . Also if q be greater than p , and therefore (in a duplicate ratio) q^2 will be greater than p^2 . So that (*cæteris paribus*) the degree of Convergency is here to be estimated, from the Rate according to which the Coefficients a, b, c , &c. continually decrease, compounded with the Ratio, (or rather its duplicate,) according to which q shall be suppos'd to be greater than p .

The same things obtaining as before, the Term $\frac{l p^n}{q^n}$ will be what was call'd the *Supplement* of the Series. For if the Series be continued to a number of Terms denominated by n , then instead of all the rest of the Terms *in infinitum*, we may introduce this Supplement, and then we shall have the accurate value of a , instead of an approximation to that value. Here the first Sign is to be taken if n is an odd number, and the other when it is even. Thus if $n = 1$, and consequently $k = a$, and $l = b$, we shall have $\frac{bp + aq}{q} = a$. Or if $n = 2$, and $l = c$, then $\frac{bp + aq}{q} - \frac{cp + bq}{q} \times \frac{p}{q} + \frac{cp^2}{q^2} = a$. Or if $n = 3$, $l = d$, then $\frac{bp + aq}{q} - \frac{cp + bq}{q} \times \frac{p}{q} + \frac{dp + cq}{q} \times \frac{p^2}{q^2} - \frac{dp^3}{q^3} = a$. And so on. Here the taking in of the Supplement always compleats the value of a , and makes it perfect, whether the Series be converging or diverging; which will always be the best way of proceeding, when that Supplement can readily be known. But as this rarely happens, in such infinite Series as generally occur, we must have recourse to infinite converging Series, wherein this Supplement, as well as the Terms of the Series, are infinitely diminish'd; and therefore after a competent number of them are collected, the rest may be all neglected *in infinitum*.

From this general Series, the better to assist the Imagination, we will descend to a few particular Instances of converging Series in pure Numbers. Let the Coefficients a, b, c, d , &c. be expounded by $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c. respectively; then $\frac{\frac{1}{2}p + q}{q} - \frac{\frac{1}{3}p + \frac{1}{2}q}{q} \times \frac{p}{q} + \frac{\frac{1}{4}p + \frac{1}{3}q}{q} \times \frac{p^2}{q^2}$, &c. $= 1$, or $\frac{p + 2q}{2q} - \frac{2p + 3q}{2 \times 3q} \times \frac{p}{q} + \frac{3p + 4q}{3 \times 4q} \times \frac{p^2}{q^2} - \frac{4p + 5q}{4 \times 5q} \times \frac{p^3}{q^3}$, &c. $= 1$. That the Series hence arising may converge, make p less than

than q in any given ratio, suppose $\frac{p}{q} = \frac{1}{2}$, or $p = 1$, $q = 2$, then $\frac{5}{4} - \frac{2}{3} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4} - \frac{7}{8} \times \frac{1}{8}$, &c. $= 1$. That is, this Series of Fractions, which is computed by Binary Arithmetick, or by the Reciprocals of the Powers of Two, if infinitely continued will finally be equal to Unity. Or if we desire to stop at these four Terms, and instead of the rest *ad infinitum* if we would introduce the Supplement which is equivalent to them, and which is here known to be $\frac{1}{3} \times \frac{1}{16}$, or $\frac{1}{48}$, we shall have $\frac{5}{4} - \frac{2}{3} + \frac{1}{2} - \frac{7}{8} + \frac{1}{48} = 1$, as is easy to prove. Or let the same Coefficients be expounded by 1, $-\frac{1}{2}$, $\frac{1}{3}$, $-\frac{1}{4}$, $\frac{1}{5}$, &c. then it will be $\frac{2q-p}{2q} + \frac{3q-2p}{2 \times 3q} \times \frac{p}{q} + \frac{4q-3p}{3 \times 4q} \times \frac{p^2}{q^2} + \frac{5q-4p}{4 \times 5q} \times \frac{p^3}{q^3}$, &c. $= 1$. This Series may either be continued infinitely, or may be sum'd after any number of Terms express'd by n , by introducing the Supplement $\frac{+p^n}{n+1 \times q^n}$ instead of all the rest. Or more particularly, if we make $q = 5p$, then $\frac{9}{2 \times 5} + \frac{13}{6 \times 5^2} + \frac{17}{12 \times 5^3} + \frac{21}{20 \times 5^4} + \frac{25}{30 \times 5^5}$, &c. $= 1$, which is a Number express'd by Quinary Arithmetick. And this is easily reduced to the Decimal Scale, by writing $\frac{2}{5}$ for $\frac{1}{5}$, and reducing the Coefficients; for then it will become 0,99999, &c. $= 1$. Now if we take these five Terms, together with the Supplement, we shall have exactly $\frac{9}{2 \times 5} + \frac{13}{6 \times 5^2} + \frac{17}{12 \times 5^3} + \frac{21}{20 \times 5^4} + \frac{25}{30 \times 5^5} + \frac{1}{6 \times 5^5} = 1$. Again, if we make here $3q = 100p$, we shall have the Series $\frac{200-3}{1 \times 2} \times \frac{1}{100} + \frac{300-6}{2 \times 3} \times \frac{3}{10000} + \frac{400-9}{3 \times 4} \times \frac{9}{1000000} + \frac{500-12}{4 \times 5} \times \frac{27}{100000000}$, &c. $= 1$, which converges very fast. And if we would reduce this to the regular Decimal Scale of Arithmetick, (which is always suppos'd to be done, before any particular Problem can be said to be compleatly solved,) we must set the Terms, when decimally reduced, orderly under one another, that their Amount or Aggregate may be discover'd; and then they will stand as in the Margin. Here the Aggregate of the first five Terms is 0,99999999595, which is a near Approximation to the Amount of the whole infinite Series, or to Unity. And if, for proof-sake, we add to this the Supplement $\frac{+p^n}{n+1 \times q^n} = \frac{p^5}{6q^5}$ $= 0,00000000405$, the whole will be Unity exactly.

0,985
147
29325
6588
15795
0,99999999595
405
1,00000000000

There are also other Methods of forming converging Series, whether general or particular, which shall approximate to a known quantity, and therefore will be very proper to explain the nature of Convergency, and to shew how the Supplement is to be introduced, when it can be done, in order to make the Series finite; which of late has been call'd the Summing of a Series. Let $A, B, C, D, E, \&c.$ and $a, b, c, d, e, \&c.$ be any two Progreffions of Terms, of which A is to be express'd by a Series, either finite or infinite, compos'd of itself and the other Terms. Suppose therefore the first Term of the Series to be a , and that p is the supplement to the value of a . Then is $A = a + p$, or $p = \frac{A-a}{1}$. As this is the whole Supplement, in order to form a Series, I shall only take such a part of it as is denominated by the Fraction $\frac{b}{B}$, and put q for the second Supplement. That is, I will assume $\frac{A-a}{1} = (p =) \frac{A-a}{1} \times \frac{b}{B} + q$, or $q = \left(\frac{A-a}{1} \times 1 - \frac{b}{B} \right) \frac{A-a}{B} \times \frac{B-b}{1}$. Again, as this is the whole value of the Supplement q , I shall only assume such a part of it as is denominated by the Fraction $\frac{c}{C}$, and for the next Supplement put r . That is, $\frac{A-a}{B} \times \frac{B-b}{1} = (q =) \frac{A-a}{B} \times \frac{B-b}{C} c + r$, or $r = \left(\frac{A-a}{B} \times \frac{B-b}{1} \times 1 - \frac{c}{C} \right) \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1}$. Now as this is the whole value of the Supplement r , I only assume such a part of it as is denominated by the Fraction $\frac{d}{D}$, and for the next Supplement put s . That is, $\frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1} = (r =) \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + s$, or $s = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{1} \times 1 - \frac{d}{D} = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{1}$. And so on as far as we please. So that at last we have the value of $A = a + p$, where the Supplement $p = \frac{A-a}{B} b + q$, where the second Supplement $q = \frac{A-a}{B} \times \frac{B-b}{C} c + r$, where $r = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + s$, where $s = \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{E} e + t$. And so on *ad infinitum*. That is finally $A = a + \frac{A-a}{B} b + \frac{A-a}{B} \times \frac{B-b}{C} c + \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} d + \frac{A-a}{B} \times \frac{B-b}{C} \times \frac{C-c}{D} \times \frac{D-d}{E} e, \&c.$ where $A, B, C, D, E, \&c.$ and $a, b, c, d, e, \&c.$ may be any two Progreffions of Numbers whatever, whether regular or desultory, ascending or descending. And when it

it happens in these Progressions, that either $A = a$, or $B = b$, or $C = c$, &c. then the Series terminates of itself, and exhibits the value of A in a finite number of Terms: But in other cases it approximates indefinitely to the value of A . But in the case of an infinite Approximation, the said Progressions ought to proceed regularly, according to some stated Law. Here it will be easy to observe, that if K and k are put to represent any two Terms indefinitely in the aforesaid Progressions, whose places are denoted by the number n , and if L and l are the Terms immediately following; then the Term in the Series denoted by $n + 1$ will be form'd from the preceding Term, by multiplying it by $\frac{K - k}{kL} l$. As if $n = 1$, then $K = A$, $k = a$, $L = B$, $l = b$, and the second Term will be $a + \frac{A - a}{aB} b = \frac{A - a}{B} b$. If $n = 2$, then $K = B$, $k = b$, $L = C$, $l = c$, and the third Term will be $\frac{A - a}{B} b \times \frac{B - b}{bC} c = \frac{A - a}{B} \times \frac{B - b}{C} c$; and so of the rest. And whenever it shall happen that $L = l$, then the Series will stop at this Term, and proceed no farther. And the Series approximates so much the faster, *cæteris paribus*, as the Numbers A, B, C, D , &c. and a, b, c, d , &c. approach nearer to each other respectively.

Now to give some Examples in pure Numbers. Let A, B, C, D , &c. $= 2, 2, 2, 2$, &c. and a, b, c, d , &c. $= 1, 1, 1, 1$, &c. then we shall have $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, &c. And so always, when the given Progressions are Ranks of equals, the Series will be a Geometrical Progression. If we would have this Progression stop at the next Term, we may either suppose the first given Progression to be $2, 2, 2, 2, 2, 1$, or the second to be $1, 1, 1, 1, 1, 2$, 'tis all one which. For in either case we shall have $L = l$, that is $F = f$, and therefore the last Term must be multiply'd by $\frac{K - k}{k}$, or $\frac{E - e}{e} = 1$.

Then the Progression or Series becomes $2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$. Again, if A, B, C, D , &c. $= 5, 5, 5, 5$, &c. and a, b, c, d , &c. $= 4, 4, 4, 4$, &c. then $5 = 4 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}$, &c. or $\frac{1}{4} = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4}$, &c. Or if A, B, C, D , &c. $= 4, 4, 4, 4$, &c. and a, b, c, d , &c. $= 5, 5, 5, 5$, &c. then $4 = 5 - \frac{1}{5} + \frac{1}{5^2} - \frac{1}{5^3} + \frac{1}{5^4}$, &c. If A, B, C, D , &c. $= 5, 5, 5, 5$, &c. and a, b, c, d , &c. $= 6, 7, 8, 9$, &c. then $5 = 6 - \frac{1}{3}7 + \frac{1}{3} \times \frac{2}{3}8 - \frac{1}{3} \times \frac{2}{3} \times \frac{3}{3}9 + \frac{1}{3} \times \frac{2}{3} \times \frac{3}{3} \times \frac{4}{3}10$, &c. If we would have the Series stop here, or if we would find one more Term, or Supplement, which should be equivalent to all the rest *ad infinitum*, (which indeed

deed might be desirable here, and in such cases as this, because of the slow Convergency, or rather Divergency of the Series,) suppose $F=f$, and therefore $\frac{E-e}{e} = \frac{5-10}{10} = -\frac{1}{2}$ must be multiply'd by the last Term. So that the Series becomes $5 = 6 - \frac{1}{5}7 + \frac{1}{5} \times \frac{2}{5}8 - \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5}9 + \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{4}{5}10 - \frac{1}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{4}{5}5$. If $A, B, C, D, \&c. = 2, 3, 4, 5, \&c.$ and $a, b, c, d, \&c. = 1, 2, 3, 4, \&c.$ then $2 = 1 + \frac{1}{3}2 + \frac{1}{3} \times \frac{1}{4}3 + \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}4 + \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \frac{1}{6}5, \&c.$ If $A, B, C, D, \&c. = 1, 2, 3, 4, \&c.$ and $a, b, c, d, \&c. = 2, 3, 4, 5, \&c.$ then $1 = 2 - \frac{1}{2}3 + \frac{1}{2} \times \frac{1}{3}4 - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}5 + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}6, \&c.$ And from this general Series may infinite other particular Series be easily derived, which shall perpetually converge to given Quantities; the chief use of which Speculation, I think, will be, to shew us the nature of Convergency in general.

There are many other such like general Series that may be readily form'd, which shall converge to a given Number. As if I would construct a Series that shall converge to Unity, I set down 1, together with a Rank of Fractions, both negative and affirmative, as here follows.

$$\begin{array}{r}
 1 - \frac{a}{A} - \frac{b}{B} - \frac{c}{C} - \frac{d}{D} - \frac{e}{E}, \&c. \\
 + \frac{a}{A} + \frac{b}{B} + \frac{c}{C} + \frac{d}{D} + \frac{e}{E}, \&c. \\
 \hline
 \frac{A+a}{A} + \frac{Ab-Ba}{AB} + \frac{Bc-Cb}{BC} + \frac{Cd-Dc}{CD} + \frac{De-Ed}{DE}, \&c. = 1.
 \end{array}$$

Then proceeding obliquely, I collect the Terms of each Series together, by adding the two first, then the two second, and so on. So that the whole Series thus constructed must necessarily be equal to Unity; which also is manifest by a bare Inspection of the Series. From this Series it is easy to descend to any number of particular Cases. As if we make $A, B, C, D, \&c. = 2, 3, 4, 5, \&c.$ and $a, b, c, d, \&c. = 1, 1, 1, 1, \&c.$ then $\frac{3}{2} - \frac{1}{2 \times 3} - \frac{1}{3 \times 4} - \frac{1}{4 \times 5} - \frac{1}{5 \times 6}, \&c. = 1$. Or $\frac{1}{2} = \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6}, \&c.$ And so in all other Cases. The Series will stop at a finite number of Terms, whensoever you omit to take in the first part of the Numerator of any Term. As here $\frac{3}{2} - \frac{1}{2 \times 3} - \frac{1}{3 \times 4} - \frac{1}{4 \times 5} - \frac{1}{5 \times 6} - \frac{1}{6} = 1$.

Lastly, to construct one more Series of this kind, which shall converge to Unity; I set down 1, with a Rank of Fractions along with

with it, both affirmative and negative, such as are seen here below ; which being added together obliquely as before, will produce the following Series.

$$1 + \frac{a}{A} + \frac{ab}{AB} + \frac{abc}{ABC} + \frac{abcd}{ABCD} + \frac{abcde}{ABCDE}, \&c.$$

$$- \frac{a}{A} - \frac{ab}{AB} - \frac{abc}{ABC} - \frac{abcd}{ABCD} - \frac{abcde}{ABCDE}, \&c.$$

$$\frac{A-a}{A} + \frac{B-b}{AB}a + \frac{C-c}{ABC}ab + \frac{D-d}{ABCD}abc + \frac{E-e}{ABCDE}abcd, \&c. = 1.$$

This Series may be made to stop at any finite number of Terms, if you omit to take in the latter part of the Binomial in any Term. Or you may derive particular Series from it, which shall have any Rate of Convergency.

For an Example of this Series, make $A, B, C, D, \&c. = 3, 3, 3, 3, \&c.$ and $a, b, c, d, \&c. = 1, 1, 1, 1, \&c.$ then $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}, \&c. = 1$, or $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}, \&c. = \frac{1}{2}$. And whenever $A, B, C, \&c.$ and $a, b, c, \&c.$ are Ranks of Equals, the Series will be a Geometrical Progression.

Again, make $A, B, C, D, \&c. = 2, 3, 4, 5, \&c.$ and $a, b, c, d, \&c. = 1, 1, 1, 1, \&c.$ then $\frac{1}{2} + \frac{2}{2 \times 3} + \frac{3}{2 \times 3 \times 4} + \frac{4}{2 \times 3 \times 4 \times 5} + \frac{5}{2 \times 3 \times 4 \times 5 \times 6}, \&c. = 1$. Or in a finite number of Terms $\frac{1}{2} + \frac{1}{3} + \frac{1}{2 \times 4} + \frac{1}{2 \times 3 \times 5} + \frac{1}{2 \times 3 \times 4 \times 5} = 1$. And the like may be observed of others in an infinite variety.

And thus having prepared the way for what follows, by explaining the nature of infinite Series in general, by discovering their origin and manner of convergency, and by shewing their connexion with our common Arithmetick ; I shall now return to our Author's Methods of Operation, or to the Reduction of compound Quantities to such infinite Series.

SECT. II. *The Resolution of simple Equations, or pure Powers, by Infinite Series.*

3, 4. **T**HE Author begins his Reduction of compound Quantities, to an equivalent infinite Series of simple Terms, first by shewing how the Process may be perform'd in Division. Now in his Example the manner of the Operation is thus, in imitation

tation of the usual praxis of Division in Numbers. In order to obtain the Quotient of aa divided by $b+x$, or to resolve the compound Fraction $\frac{aa}{b+x}$ into a Series of simple Terms, first find the Quotient of aa divided by b , the first Term of the Divisor. This is $\frac{aa}{b}$, which write in the Quote. Then multiply the Divisor by this Term, and set the Product $aa + \frac{aax}{b}$ under the Dividend, from whence it must be subtracted, and will leave the Remainder $-\frac{aax}{b}$. Then to find the next Term (or Figure) of the Quotient, divide the Remainder by the first Term of the Divisor, or by b , and put the Quotient $-\frac{aax}{b^2}$ for the second Term of the Quote. Multiply the Divisor by this second Term, and the Product $-\frac{aax}{b} - \frac{aaxx}{bb}$ set orderly under the last Remainder; from whence it must be subtracted, to find the new Remainder $+\frac{aaxx}{bb}$. Then to find the next Term of the Quotient, you are to proceed with this new Remainder as with the former; and so on *in infinitum*. The Quotient therefore is $\frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4}$, &c. (or $\frac{a^2}{b}$ into $1 - \frac{x}{b} + \frac{x^2}{b^2} - \frac{x^3}{b^3}$, &c.) So that by this Operation the Number or Quantity $\frac{aa}{b+x}$, (or $a^2 \times \overline{b+x}^{-1}$) is reduced from that Scale in Arithmetick whose Root is $b+x$, to an equivalent Number, the Root of whose Scale, (or whose converging quantity) is $\frac{x}{b}$. And this Number, or infinite Series thus found, will converge so much the faster to the truth, as b is greater than x .

To apply this, by way of illustration, to an instance or two in common Numbers. Suppose we had the Fraction $\frac{1}{7}$, and would reduce it from the septenary Scale, in which it now appears, to an equivalent Series, that shall converge by the Powers of 6. Then we shall have $\frac{1}{7} = \frac{1}{6+1}$; and therefore in the foregoing general Fraction $\frac{aa}{b+x}$, make $a=1$, $b=6$, and $x=1$, and the Series will become $\frac{1}{6} - \frac{1}{6^2} + \frac{1}{6^3} - \frac{1}{6^4}$, &c. which will be equivalent to $\frac{1}{7}$. Or if we would reduce it to a Series converging by the Powers of 8, because $\frac{1}{7} = \frac{1}{8-1}$, make $a=1$, $b=8$, and $x=-1$, then

then $\frac{1}{7} = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \frac{1}{8^4}$, &c. which Series will converge faster than the former. Or if we would reduce it to the common Denary (or Decimal) Scale, because $\frac{1}{7} = \frac{1}{10 - 3}$, make $a = 1$, $b = 10$, and $x = -3$; then $\frac{1}{7} = \frac{1}{10} + \frac{3}{10^2} + \frac{9}{10^3} + \frac{27}{10^4} + \frac{81}{10^5}$, &c. $= 0,1428$, &c. as may be easily collected. And hence we may observe, that this or any other Fraction may be reduced a great variety of ways to infinite Series; but that Series will converge fastest to the truth, in which b shall be greatest in respect of x . But that Series will be most easily reduced to the common Arithmetick, which converges by the Powers of 10, or its Multiples. If we should here resolve 7 into the parts $3 + 4$, or $2 + 5$, or $1 + 6$, &c. instead of converging we should have diverging Series, or such as require a Supplement to be taken in.

And we may here farther observe, that as in Division of common Numbers, we may stop the process of Division whenever we please, and instead of all the rest of the Figures (or Terms) *ad infinitum*, we may write the Remainder as a Numerator, and the Divisor as the Denominator of a Fraction, which Fraction will be the Supplement to the Quotient: so the same will obtain in the Division of Species. Thus in the present Example, if we will stop

at the first Term of the Quotient, we shall have $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b \times b + x}$.

Or if we will stop at the second Term, then $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2} +$

$\frac{a^2x^2}{b^3+b^2x}$. Or if we will stop at the third Term, then $\frac{aa}{b+x} = \frac{aa}{b} -$

$\frac{aax}{b^2} + \frac{aax^2}{b^3} - \frac{a^2x^3}{b^4+b^3x}$. And so in the succeeding Terms, in which

these Supplements may always be introduced, to make the Quotient compleat. This Observation will be found of good use in some of the following Speculations, when a complicate Fraction is not to be intirely resolved, but only to be depresso'd, or to be reduced to a simpler and more commodious form.

Or we may hence change Division into Multiplication. For having found the first Term of the Quotient, and its Supplement, or the Equation $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2+bx}$; multiplying it by $\frac{x}{b}$, we shall

have $\frac{aax}{b^2+bx} = \frac{aax}{b^2} - \frac{a^2x^2}{b^3+b^2x}$, so that substituting this value of

$\frac{aax}{b^2+bx}$ in the first Equation, it will become $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2} +$

$\frac{a^2x^2}{b^3+b^2x}$, where the two first Terms of the Quotient are now known.

Multiply this by $\frac{x^2}{b^2}$, and it will become $\frac{a^2x^2}{b^3 + b^2x} = \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5 + b^4x}$, which being substituted in the last Equation, it will become $\frac{aa}{b+x} = \frac{aa}{b} - \frac{aax}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5 + b^4x}$, where the four first Terms of the Quotient are now known. Again, multiply this Equation by $\frac{x^4}{b^4}$, and it will become $\frac{aax^4}{b^5 + b^4x} = \frac{a^2x^4}{b^5} - \frac{a^2x^5}{b^6} + \frac{a^2x^6}{b^7} - \frac{a^2x^7}{b^8} + \frac{a^2x^8}{b^9 + b^8x}$, which being substituted in the last Equation, it will become $\frac{aa}{b+x} = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5} - \frac{a^2x^5}{b^6} + \frac{a^2x^6}{b^7} - \frac{a^2x^7}{b^8} + \frac{a^2x^8}{b^9 + b^8x}$, where eight of the first Terms are now known. And so every succeeding Operation will double the number of Terms, that were before found in the Quotient.

This method of Reduction may be thus very conveniently imitated in Numbers, or we may thus change Division into Multiplication. Suppose (for instance) I would find the Reciprocal of the Prime Number 29, or the value of the Fraction $\frac{1}{29}$ in Decimal Numbers. I divide 1,0000, &c. by 29, in the common way, so far as to find two or three of the first Figures, or till the Remainder becomes a single Figure, and then I assume the Supplement to compleat the Quotient. Thus I shall have $\frac{1}{29} = 0,03448\frac{8}{29}$ for the compleat Quotient, which Equation if I multiply by the Numerator 8, it will give $\frac{8}{29} = 0,27584\frac{64}{29}$, or rather $\frac{8}{29} = 0,27586\frac{6}{29}$. I substitute this instead of the Fraction in the first Equation, and I shall have $\frac{1}{29} = 0,0344827586\frac{6}{29}$. Again, I multiply this Equation by 6, and it will give $\frac{6}{29} = 0,2068965517\frac{7}{29}$, and then by Substitution $\frac{1}{29} = 0,03448275862068965517\frac{7}{29}$. Again, I multiply this Equation by 7, and it becomes $\frac{7}{29} = 0,24137931034482758620\frac{2}{29}$, and then by Substitution $\frac{1}{29} = 0,0344827586206896551724137931034482758620\frac{2}{29}$, where every Operation will at least double the number of Figures found by the preceding Operation. And this will be an easy Expedient for converting Division into Multiplication in all Cases. For the Reciprocal of the Divisor being thus found, it may be multiply'd into the Dividend to produce the Quotient.

Now as it is here found, that $\frac{aa}{b+x} = \frac{aa}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4}$, &c. which Series will converge when b is greater than x ; so when it happens to be otherwise, or when x is greater than b , that the Powers of x may be in the Denominators we must have recourse to the

the other Case of Division, in which we shall find $\frac{aa}{x+b} = \frac{aa}{x} - \frac{a^2b}{x^2} + \frac{a^2b^2}{x^3} - \frac{a^2b^3}{x^4}$, &c. and where the Division is perform'd as before.

5, 6. In these Examples of our Author, the Process of Division (for the exercise of the Learner) may be thus exhibited :

$$\begin{array}{r}
 1+x^2 \overline{) 1+x^2+0(1-x^2+x^4-x^6+x^8, \&c.} \quad 1+x^{\frac{1}{2}}-3x \overline{) 2x^{\frac{1}{2}}-x^{\frac{3}{2}}(2x^{\frac{1}{2}}-2x+7x^{\frac{3}{2}}-13x^{\frac{5}{2}}, \&c.} \\
 \underline{1+x^2} \\
 0 \\
 \underline{-x^2-x^4} \\
 0 \\
 \underline{+x^4+x^6} \\
 0 \\
 \underline{-x^6-x^8} \\
 0 \\
 \underline{+x^8}
 \end{array}$$

Now in order to a due Convergency, in each of these Examples, we must suppose x to be less than Unity; and if x be greater than Unity, we must invert the Terms, and then we shall have $\frac{1}{xx+1}$

$$= \frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{x^8}, \&c. \quad \text{and} \quad \frac{-x^{\frac{1}{2}}+2x^{\frac{3}{2}}}{-3x+x^{\frac{1}{2}}+1} = \frac{1}{3}x^{\frac{1}{2}} + \frac{1}{9} - \frac{14}{27x^{\frac{1}{2}}} - \frac{11}{81x}, \&c.$$

7, 8, 9, 10. This Notation of Powers and Roots by integral and fractional, affirmative and negative, general and particular Indices, was certainly a very happy Thought, and an admirable Improvement of Analyticks, by which the practice is render'd easy, regular, and universal. It was chiefly owing to our Author, at least he carried on the Analogy, and made it more general. A Learner should be well acquainted with this Notation, and the Rules of its several Operations should be very familiar to him, or otherwise he will often find himself involved in difficulties. I shall not enter into any farther discussion of it here, as not properly belonging to this place, or subject, but rather to the vulgar Algebra.

11. The Author proceeds to the Extraction of the Roots of pure Equations, which he thus performs, in imitation of the usual Process in Numbers. To extract the Square-root of $aa+xx$; first the Root of aa is a , which must be put in the Quote. Then the Square of this, or aa , being subtracted from the given Power, leaves $+xx$ for a Resolvend. Divide this by twice the Root, or $2a$, which is

the first part of the Divisor, and the Quotient $\frac{xx}{2a}$ must be made the second Term of the Root, as also the second Term of the Divisor. Multiply the Divisor thus compleated, or $2a + \frac{xx}{2a}$, by the second Term of the Root, and the Product $xx + \frac{x^4}{4aa}$ must be subtracted from the Resolvend. This will leave $-\frac{x^4}{4a^2}$ for a new Resolvend, which being divided by the first Term of the double Root, or $2a$, will give $-\frac{x^4}{8a^3}$ for the third Term of the Root. Twice the Root before found, with this Term added to it, or $2a + \frac{x^2}{a} - \frac{x^4}{8a^3}$, being multiply'd by this Term, the Product $-\frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{64a^6}$ must be subtracted from the last Resolvend, and the Remainder $+\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$ will be a new Resolvend, to be proceeded with as before, for finding the next Term of the Root; and so on as far as you please. So that we shall have $\sqrt{aa + xx} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$, &c.

It is easy to observe from hence, that in the Operation every new Column will give a new Term in the Quote or Root; and therefore no more Columns need be form'd than it is intended there shall be Terms in the Root. Or when any number of Terms are thus extracted, as many more may be found by Division only. Thus having found the three first Terms of the Root $a + \frac{x^2}{2a} - \frac{x^4}{8a^3}$, by their double $2a + \frac{x^2}{a} - \frac{x^4}{4a^3}$, dividing the third Remainder or Resolvend $+\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$, the three first Terms of the Quotient $\frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$ will be the three succeeding Terms of the Root.

The Series $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5}$, &c. thus found for the square-root of the irrational quantity $aa + xx$, is to be understood in the following manner. In order to a due convergency a is to be suppos'd greater than x , that the Root or converging quantity $\frac{x}{a}$ may be less than Unity, and that a may be a near approximation to the square-root required. But as this is too little, it is encreased by the small quantity $\frac{x^2}{2a}$, which now makes it too big. Then by the next

Operation

Operation it is diminish'd by the still smaller quantity $\frac{x^4}{8a^3}$; which diminution being too much, it is again encreas'd by the very small quantity $\frac{x^6}{16a^5}$, which makes it too great, in order to be farther diminish'd by the next Term. And thus it proceeds *in infinitum*, the Augmentations and Diminutions continually correcting one another, till at last they become inconsiderable, and till the Series (so far continued) is a sufficiently near Approximation to the Root required.

12. When a is less than x , the order of the Terms must be inverted, or the square-root of $xx + aa$ must be extracted as before; in which case it will be $x + \frac{aa}{2x} - \frac{a^4}{8x^3}$, &c. And in this Series the converging quantity, or the Root of the Scale, will be $\frac{a}{x}$. These two Series are by no means to be understood as the two different Roots of the quantity $aa + xx$; for each of the two Series will exhibit those two Roots, by only changing the Signs. But they are accommodated to the two Cases of Convergency, according as a or x may happen to be the greater quantity.

I shall here resolve the foregoing Quantity after another manner, the better to prepare the way for what is to follow. Suppose then $yy = aa + xx$, where we may find the value of the Root y by the following Process: $yy = aa + xx =$ (if $y = a + p$) $aa + 2ap + pp$; or $2ap + pp = xx =$ (if $p = \frac{xx}{2a} + q$) $xx + 2aq + \frac{x^4}{4a^2} + \frac{x^2q}{a} + qq$; or $2aq + \frac{x^2q}{a} + qq = -\frac{x^4}{4a^2} =$ (if $q = -\frac{x^4}{8a^3} + r$) $-\frac{x^4}{4a^2} + 2ar - \frac{x^6}{8a^4} + \frac{x^2r}{a} + \frac{x^8}{64a^6} - \frac{x^4r}{4a^3} + r^2$; or $2ar + \frac{xxr}{a} - \frac{x^4r}{4a^3} + rr = \frac{x^6}{8a^4} - \frac{x^8}{64a^6} =$ (if $r = \frac{x^6}{16a^5} + s$) &c. which Process may be thus explain'd in words.

In order to find $\sqrt{aa + xx}$, or the Root y of this Equation $yy = aa + xx$, suppose $y = a + p$, where a is to be understood as a pretty near Approximation to the value of y , (the nearer the better,) and p is the small Supplement to that, or the quantity which makes it compleat. Then by Substitution is derived the first Supplemental Equation $2ap + pp = xx$, whose Root p is to be found. Now as $2ap$ is much bigger than pp , (for $2a$ is bigger than the Supplement p ;) we shall have nearly $p = \frac{xx}{2a}$, or at least we shall have exactly $p = \frac{xx}{2a} + q$, supposing q to represent the second Supplement

ment of the Root. Then by Substitution $2aq + \frac{xx}{a}q + qq = -\frac{x^4}{4a^2}$ will be the second Supplemental Equation, whose Root q is the second Supplement. Therefore $\frac{xx}{a}q$ will be a little quantity, and qq much less, so that we shall have nearly $q = -\frac{x^4}{8a^3}$, or accurately $q = -\frac{x^4}{8a^3} + r$, if r be made the third Supplement to the Root. And therefore $2ar + \frac{xx}{a}r - \frac{x^4}{4a^3}r + r^2 = \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$ will be the third Supplemental Equation, whose Root is r . And thus we may go on as far as we please, to form Residual or Supplemental Equations, whose Roots will continually grow less and less, and therefore will make nearer and nearer Approaches to the Root y , to which they always converge. For $y = a + p$, where p is the Root of this Equation $2ap + pp = xx$. Or $y = a + \frac{xx}{2a} + q$, where q is the Root of this Equation $2aq + \frac{xx}{a}q + qq = -\frac{x^4}{4a^2}$. Or $y = a + \frac{xx}{2a} - \frac{x^4}{8a^3} + r$, where r is the Root of this Equation $2ar + \frac{xx}{a}r - \frac{x^4}{4a^3}rr = \frac{x^6}{8a^4} - \frac{x^8}{64a^6}$. And so on. The Resolution of any one of these Quadratick Equations, in the ordinary way, will give the respective Supplement, which will compleat the value of y .

I took notice before, upon the Article of Division, of what may be call'd a Comparison of Quotients; or that one Quotient may be exhibited by the help of another, together with a Series of known or simple Terms. Here we have an Instance of a like Comparison of Roots; or that the Root of one Equation may be express'd by the Root of another, together with a Series of known or simple Terms, which will hold good in all Equations whatever. And to carry on the Analogy, we shall hereafter find a like Comparison of Fluents; where one Fluent, (suppose, for instance, a Curvilinear Area,) will be express'd by another Fluent, together with a Series of simple Terms. This I thought fit to insinuate here, by way of anticipation, that I might shew the constant uniformity and harmony of Nature, in these Speculations, when they are duly and regularly pursued.

But I shall here give, *ex abundanti*, another Method for this, and such kind of Extractions, tho' perhaps it may more properly belong to the Resolution of Affected Equations, which is soon to follow; however it may serve as an Introduction to their Solution.

The first Residual or Supplemental Equation in the foregoing Process was $2ap + pp = xx$, which may be resolved in this manner.

Because $p = \frac{xx}{2a + p}$, it will be by Division $p = \frac{xx}{2a} - \frac{x^2 p}{4a^2} + \frac{x^2 p^2}{8a^3} - \frac{x^2 p^3}{16a^4} + \frac{x^2 p^4}{32a^5}$, &c. Divide all the Terms of this Series (except the first) by p , and then multiply them by the whole Series, or by the value of p , and you will have $p = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^4 p}{8a^4} - \frac{3x^4 p^2}{32a^5} + \frac{x^4 p^3}{16a^6}$, &c. where the two first Terms are clear'd of p . Divide all the Terms of this Series, except the two first, by p , and multiply them by the value of p , or by the first Series, and you will have a Series for p in which the three first Terms are clear'd of p . And by repeating the Operation, you may clear as many Terms of p as you please. So that at last you will have $p = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}$, &c. which will give the same value of y as before.

13, 14, 15, 16, 17, 18. The several Roots of these Examples, and of all other pure Powers, whether they are Binomials, Trinomials, or any other Multinomials, may be extracted by pursuing the Method of the foregoing Process, or by imitating the like *Praxes* in Numbers. But they may be perform'd much more readily by general Theorems computed for that purpose. And as there will be frequent occasion, in the ensuing Treatise, for certain general Operations to be perform'd with infinite Series, such as Multiplication, Division, raising of Powers, and extracting of Roots; I shall here derive some Theorems for those purposes.

I. Let $A + B + C + D + E$, &c. $P + Q + R + S + T$, &c. and $\alpha + \beta + \gamma + \delta + \epsilon$, &c. represent the Terms of three several Series respectively, and let $A + B + C + D + E$, &c. into $P + Q + R + S + T$, &c. $= \alpha + \beta + \gamma + \delta + \epsilon$, &c. Then by the known Rules of Multiplication, by which every Term of one Factor is to be multiply'd into every Term of the other, it will be $\alpha = AP$, $\beta = AQ + BP$, $\gamma = AR + BQ + CP$, $\delta = AS + BR + CQ + DP$, $\epsilon = AT + BS + CR + DQ + EP$; and so on. Then by Substitution it will be

$$A + B + C + D + E, \text{ &c.} \times P + Q + R + S + T, \text{ &c.} = AP + BP + CP + DP + EP, \text{ &c.} \\ + AQ + BQ + CQ + DQ \\ + AR + BR + CR \\ + AS + BS \\ + AT$$

And

And this will be a ready Theorem for the Multiplication of any infinite Series into each other; as in the following Example.

$$\begin{array}{cccccccccc}
 \text{(A)} & \text{(B)} & \text{(C)} & \text{(D)} & \text{(E)} & \text{(P)} & \text{(Q)} & \text{(R)} & \text{(S)} & \text{(T)} \\
 a + \frac{1}{2}x + \frac{x^2}{3a} + \frac{x^3}{4a^2} + \frac{x^4}{5a^3}, & \&c. & \text{into} & a - \frac{1}{3}x + \frac{x^2}{5a} - \frac{x^3}{7a^2} + \frac{x^4}{9a^3}, & \&c. \\
 = a^2 + x\frac{1}{2}a + \frac{1}{3}x^2 + \frac{x^3}{4a} + \frac{x^4}{5a^2}, & \&c. & = a^2 + \frac{1}{6}ax + \frac{1}{30}x^2 + \frac{121x^3}{1260a^2} + \frac{281x^4}{1260a^3}, & \&c. \\
 - \frac{1}{3}ax - \frac{1}{6}x^2 - \frac{x^3}{9a} - \frac{x^4}{12a^2} \\
 + \frac{1}{5}x^2 + \frac{x^3}{10a} + \frac{x^4}{15a^2} \\
 - \frac{x^3}{7a} - \frac{x^4}{14a^2} \\
 + \frac{x^4}{9a^2}
 \end{array}$$

And so in all other cases.

II. From the same Equations above we shall have $A = \frac{a}{P}$, $B = \frac{\beta - AQ}{P}$, $C = \frac{\gamma - BQ - AR}{P}$, $D = \frac{\delta - CQ - BR - AS}{P}$, $E = \frac{\epsilon - DQ - CR - BS - AT}{P}$, &c. And then by Substitution $\frac{a+\beta+\gamma+\delta+\epsilon, \&c.}{P+Q+R+S+T, \&c.}$
 $= (A+B+C+D+E, \&c.) = \frac{a}{P} + \frac{\beta - AQ}{P} + \frac{\gamma - BQ - AR}{P} + \frac{\delta - CQ - BR - AS}{P} + \frac{\epsilon - DQ - CR - BS - AT}{P}$, &c. This Theorem will serve commodiously for the Division of one infinite Series by another. Here for conveniency sake the Capitals A, B, C, D, &c. are retained in the Theorem, to denote the first, second, third, fourth, &c. Terms of the Series respectively.

Thus, for Example, if we would divide the Series $\frac{(a)}{a^2} + \frac{(\beta)}{\frac{1}{2}ax} + \frac{(\gamma)}{\frac{1}{30}x^2} + \frac{(\delta)}{\frac{121x^3}{1260a}} + \frac{(\epsilon)}{\frac{281x^4}{1260a^2}}, \&c.$ by the Series $a + \frac{1}{2}x + \frac{x^2}{3a} + \frac{x^3}{4a^2} + \frac{x^4}{5a^3}, \&c.$ the Quotient will be $a + \frac{\frac{1}{2}ax - \frac{1}{2}aA}{a} + \frac{\frac{1}{30}x^2 - \frac{x^2}{3a}A - \frac{1}{2}xB}{a} + \frac{\frac{121x^3}{1260a^2} - \frac{x^3}{4a^2}A - \frac{x^2}{3a}B - \frac{1}{2}xC}{a}$, &c. Or restoring the Values of A, B, C, D, &c. which represent the several Terms as they stand in order, the Quotient will become $a - \frac{1}{3}x + \frac{x^2}{5a} - \frac{x^3}{7a^2} + \frac{x^4}{9a^3}, \&c.$ And after the same manner in all other Examples.

III. In the last Theorem make $\alpha=1, \beta=0, \gamma=0, \delta=0, \&c.$ then $\frac{1}{P+Q+R+S+T, \&c.} = \frac{1}{P} - \frac{AQ}{P} - \frac{BQ+AR}{P} - \frac{CQ+BR+AS}{P} - \frac{DQ+CR+BS+AT}{P}, \&c.$ which Theorem will readily find the Reciprocal of any infinite Series. Here A, B, C, D, &c. denote the several Terms of the Series in order, as before.

Thus if we would know the Reciprocal of the Series $a + \frac{1}{2}x + \frac{x^2}{3a} + \frac{x^3}{4a^2} + \frac{x^4}{5a^3}, \&c.$ we shall have by Substitution $\frac{1}{a} - \frac{\frac{1}{2}xA}{a} - \frac{\frac{x^2}{3a}A + \frac{1}{2}xB}{a} - \frac{\frac{x^3}{4a^2}A + \frac{x^2}{3a}B + \frac{1}{2}xC}{a} - \frac{\frac{x^4}{5a^3}A + \frac{x^3}{4a^2}B + \frac{x^2}{3a}C + \frac{1}{2}xD}{a},$

&c. And restoring the Values of A, B, C, D, &c. it will be $\frac{1}{a} - \frac{x}{2a^2} - \frac{x^2}{12a^3} + \frac{x^3}{8a^4} - \frac{79x^4}{720a^5}, \&c.$ for the Reciprocal required.

Ex. 2. $\frac{1}{1 - \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{16}x^3, \&c.} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{8}x^3, \&c.$ And so of others.

IV. In the first Theorem if we make $P=A, Q=B, R=C, S=D, \&c.$ that is, if we make both to be the same Series; we shall have $A+B+C+D+E+F+G, \&c. |^2 = A^2 + 2AB + 2AC + 2AD + 2AE + 2AF + 2AG, \&c. + B^2 + 2BC + 2BD + 2BE + 2BF + C^2 + 2CD + 2CE + D^2$

which will be a Theorem for finding the Square of any infinite Series.

Ex. 1. $\left| \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}, \&c. \right|^2 = \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{x^8}{16a^6} - \frac{5x^{10}}{128a^8} + \frac{7x^{12}}{256a^{10}}, \&c. + \frac{x^8}{64a^6} - \frac{x^{10}}{64a^8} + \frac{5x^{12}}{512a^{10}} + \frac{x^{12}}{256a^{10}}$
 $= \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8} + \frac{21x^{12}}{512a^{10}}, \&c.$

Ex. 2. $\left| -\frac{1}{2}x^{\frac{1}{2}} - \frac{1}{8}x^{\frac{3}{2}} - \frac{1}{16}x^{\frac{5}{2}}, \&c. \right|^2 = \frac{1}{4}x^3 + \frac{1}{8}x^4 + \frac{5}{16}x^5, \&c.$

Ex. 3. $\left| \frac{bx}{2a} - \frac{x^2}{2a} + \frac{bx^3}{4a^3}, \&c. \right|^2 = \frac{b^2x^2}{4a^2} - \frac{bx^3}{2a^2} + \frac{x^4}{4a^2}, \&c. - \frac{b^2x^2}{8a^3} + \frac{b^3x^3}{16a^5} - \frac{b^3x^3}{8a^4} + \frac{3b^2x^4}{8a^4} + \frac{17b^4x^4}{64a^4}$

Ex. 4. $\left| \frac{x^3}{3a^2} - \frac{x^6}{9a^3} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}, \&c. \right|^2 = \frac{x^6}{9a^4} - \frac{2x^9}{27a^7} + \frac{13x^{12}}{243a^{10}} - \frac{30x^{15}}{729a^{13}}, \&c.$

V. In this last Theorem, if we make $A^2 = P$, $2AB = Q$, $2AC + B^2 = R$, $2AD + 2BC = S$, $2AE + 2BD + C^2 = T$, &c. we shall have $A = P^{\frac{1}{2}}$, $B = \frac{Q}{2A}$, $C = \frac{R - B^2}{2A}$, $D = \frac{S - 2BC}{2A}$, $E = \frac{T - 2BD - C^2}{2A}$, &c. Or $P + Q + R + S + T + U$, &c. $^{\frac{1}{2}} = P^{\frac{1}{2}} + \frac{Q}{2A} + \frac{R - B^2}{2A} + \frac{S - 2BC}{2A} + \frac{T - 2BD - C^2}{2A} + \frac{U - 2BE - 2CD}{2A}$, &c. By this Theorem the Square-root of any infinite Series may easily be extracted. Here A, B, C, D, &c. will represent the several Terms of the Series as they are in succession.

$$\text{Ex. 1. } x^2 - 2ax + 2a^2 - \frac{a^3}{x} * + \frac{a^5}{4x^3}, \&c. \Big|^{\frac{1}{2}} = x - a + \frac{a^2}{2x} * - \frac{a^4}{8x^3} * \&c.$$

$$\text{Ex. 2. } \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8} + \frac{21x^{12}}{512a^{10}}, \&c. \Big|^{\frac{1}{2}} = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}, \&c.$$

VI. Because it is by the fourth Theorem $\alpha + \beta + \gamma + \delta + \epsilon, \&c. \Big|^2 = \alpha^2 + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\alpha\epsilon, \&c.$ in the third Theorem for $+ \beta^2 + 2\beta\gamma + 2\beta\delta + \gamma^2$

P, Q, R, S, T, &c. write α^2 , $2\alpha\beta$, $2\alpha\gamma + \beta^2$, $2\alpha\delta + 2\beta\gamma$, $2\alpha\epsilon + 2\beta\delta + \gamma^2$, &c. respectively. Then $\frac{1}{\alpha + \beta + \gamma + \delta + \epsilon, \&c. \Big|^2} = \frac{1}{\alpha^2} - \frac{2\alpha\beta A}{\alpha^2} + \frac{2\alpha\beta B + 2\alpha\gamma + \beta^2 \times A}{\alpha^2} - \frac{2\alpha\beta C + 2\alpha\gamma + \beta^2 \times B + 2\alpha\delta + 2\beta\gamma \times A}{\alpha^2}, \&c.$

And this will be a Theorem for finding the Reciprocal of the Square of any infinite Series. Here A, B, C, D, &c. still denote the Terms of the Series in their order.

VII. If in the first Theorem for P, Q, R, S, &c. we write A^2 , $2AB$, $2AC + B^2$, $2AD + 2BC$, &c. respectively, (that is $A + B + C + D, \&c. \Big|^2$, by Theor. 4.) we shall have $A + B + C + D + E + F, \&c. \Big|^3 = A^3 + 3A^2B + 3AB^2 + 3A^2D + 3AC^2 + 3BC^2, \&c. + 3A^2C + 6ABC + 3B^2C + 3B^2D + B^3 + 6ABD + 6ACD + 3A^2E + 6ABE + 3AF$

which will readily give the Cube of any infinite Series.

$$\text{Ex. 1. } \frac{x^3}{3a^2} - \frac{x^6}{9a^4} + \frac{5x^9}{81a^6} - \frac{10x^{12}}{243a^8}, \&c. \Big|^3 = \frac{x^9}{27a^6} - \frac{x^{12}}{27a^9} + \frac{x^{15}}{81a^{12}} - \frac{10x^{18}}{729a^{15}}, \&c. + \frac{5x^{15}}{243a^{12}} - \frac{10x^{18}}{729a^{15}} - \frac{x^{18}}{729a^{15}}$$

$$= \frac{x^9}{27a^6} - \frac{x^{12}}{27a^9} + \frac{8x^{15}}{243a^{12}} - \frac{7x^{18}}{243a^{15}}, \&c.$$

Ex. 2. $\frac{1}{2}x^2 + \frac{1}{18}x^3 + \frac{1}{216}x^4, \&c. |^{\frac{1}{3}} = \frac{1}{8}x^6 + \frac{1}{24}x^7 + \frac{1}{216}x^8, \&c.$

VIII. In the last Theorem, if we make $A^3 = P$, $3A^2B = Q$, $AB^2 + 3A^2C = R$, $B^3 + 6ABC + 3A^2D = S$, $\&c.$ then $A = P^{\frac{1}{3}}$, $B = \frac{Q}{3A^2}$, $C = \frac{R - 3AB^2}{3A^2}$, $D = \frac{S - 6ABC - B^3}{3A^2}$, $\&c.$ that is $P + Q + R + S + T, \&c. |^{\frac{1}{3}} = P^{\frac{1}{3}} + \frac{Q}{3A^2} + \frac{R - 3AB^2}{3A^2} + \frac{S - B^3 - 6ABC}{3A^2} + \frac{T - 3AC^2 - 3B^2C - 6ABD}{3A^2}, \&c.$ And by this Theorem the Cube-root of any infinite Series may be extracted. Here also $A, B, C, D, \&c.$ will represent the Terms as they stand in order.

Ex. 1. $\frac{x^9}{27a^6} - \frac{x^{12}}{27a^9} + \frac{8x^{15}}{243a^{12}} - \frac{7x^{18}}{243a^{15}}, \&c. |^{\frac{1}{3}} = \frac{x^3}{3a^2} - \frac{x^6}{9a^3} + \frac{5x^9}{81a^4} - \frac{10x^{12}}{243a^5}, \&c.$

Ex. 2. $\frac{1}{8}x^6 + \frac{1}{24}x^7 + \frac{1}{216}x^8, \&c. |^{\frac{1}{3}} = \frac{1}{2}x^2 + \frac{1}{18}x^3 + \frac{1}{216}x^4, \&c.$

IX. Because it is by the seventh Theorem $a + \beta + \gamma + \delta, \&c. |^3 = a^3 + 3a^2\beta + 3a\beta^2 + \beta^3, \&c.$ in the third Theorem for P , $+ 3a^2\gamma + 6a\beta\gamma + 3a^2\delta$

$Q, R, S, T, \&c.$ write $a^3, 3a^2\beta, 3a\beta^2 + 3a^2\gamma, \beta^3 + 6a\beta\gamma + 3a^2\delta, 3a\gamma^2 + 3\beta^2\gamma + 6a\beta\delta + 3a^2\epsilon, \&c.$ respectively; then $\frac{1}{a + \beta + \gamma + \delta, \&c. |^3} = \frac{1}{a^3} - \frac{3a^2\beta}{a^3} + \frac{3a^2\beta^2 + 3a\beta^2 + 3a^2\gamma \times A}{a^3} - \frac{3a^2\beta C + 3a\beta^2 + 3a^2\gamma \times B + B^3 + 6a\beta\gamma + 3a^2\delta \times A \&c.}{a^3}$

This Theorem will give the Reciprocal of the Cube of any infinite Series; where $A, B, C, D, \&c.$ stand for the Terms in order.

X. Lastly, in the first Theorem if we make $P = A^3$, $Q = 3A^2B$, $R = 3AB^2 + 3A^2C$, $S = B^3 + 6ABC + 3A^2D$, $\&c.$ we shall have $A + B + C + D, \&c. |^4 = A^4 + 4A^3B + 6A^2B^2 + 4AB^3, \&c.$ which $+ 4A^3C + 12A^2BC + 4A^3D$

will be a Theorem for finding the Biquadrate of any infinite Series.

And thus we might proceed to find particular Theorems for any other Powers or Roots of any infinite Series, or for their Reciprocals, or any fractional Powers compounded of these; all which will be found very convenient to have at hand, continued to a competent number of Terms, in order to facilitate the following Operations. Or it may be sufficient to lay before you the elegant and general Theorem, contrived for this purpose, by that skilful Mathematician, and my good Friend, the ingenious Mr. *A. De Moivre*, which was first publish'd in the Philosophical Transactions, N^o 230, and which will readily perform all these Operations.

Or we may have recourse to a kind of Mechanical Artifice, by which all the foregoing Operations may be perform'd in a very easy and general manner, as here follows.

When two infinite Series are to be multiply'd together, in order to find a third which is to be their Product, call one of them the Multiplicand, and the other the Multiplier. Write down upon your Paper the Terms of the Multiplicand, with their Signs, in a descending order, so that the Terms may be at equal distances, and just under one another. This you may call your fixt or right-hand Paper. Prepare another Paper, at the right-hand Edge of which write down the Terms of the Multiplier, with their proper Signs, in an ascending Order, so that the Terms may be at the same equal distances from each other as in the Multiplicand, and just over one another. This you may call your moveable or left-hand Paper. Apply your moveable Paper to your fixt Paper, so that the first Term of your Multiplier may stand over-against the first Term of your Multiplicand. Multiply these together, and write down the Product in its place, for the first Term of the Product required. Move your moveable Paper a step lower, so that two of the first Terms of the Multiplier may stand over-against two of the first Terms of the Multiplicand. Find the two Products, by multiplying each pair of the Terms together, that stand over-against one another; abbreviate them if it may be done, and set down the Result for the second Term of the Product required. Move your moveable Paper a step lower, so that three of the first Terms of the Multiplier may stand over-against three of the first Terms of the Multiplicand. Find the three Products, by multiplying each pair of the Terms together that stand over-against one another; abbreviate them, and set down the Result for the third Term of the Product. And proceed in the same manner to find the fourth, and all the following Terms.

I shall illustrate this Method by an Example of two Series, taken from the common Scale of Denary or Decimal Arithmetick; which will equally explain the Process in all other infinite Series whatever.

Let the Numbers to be multiply'd be 37,528936, &c. and 528,73041, &c. which, by supplying X or 10 where it is understood, will become the Series $3X + 7X^0 + 5X^{-1} + 2X^{-2} + 8X^{-3} + 9X^{-4} + 3X^{-5} + 6X^{-6}$, &c. and $5X^2 + 2X + 8X^0 + 7X^{-1} + 3X^{-2} + 0X^{-3} + 4X^{-4} + 1X^{-5}$, &c. and call the first the Multiplicand, and the second the Multiplier. These being disposed as is prescribed, will stand as follows.

Multiplier,

Multiplier, &c.	Multiplicand	Product	$1X^4$
$+ 1X^{-5}$	$3X$	$15X^3$	$9X^3$
$+ 4X^{-4}$	$+ 7X^0$	$+ 41X^2$	$8X^2$
$+ 0X^{-3}$	$+ 5X^{-1}$	$- + 63X$	$4X$
$+ 3X^{-2}$	$+ 2X^{-2}$	$- - + 97X^0$	$2X^0$
$+ 7X^{-1}$	$+ 8X^{-3}$	$- - + 142X^{-1}$	$6X^{-1}$
$+ 8X^0$	$+ 9X^{-4}$	$- - - + 133X^{-2}$	$8X^{-2}$
$+ 2X$	$+ 3X^{-5}$	$- - - + 138X^{-3}$	$8X^{-3}$
$5X^2$	$+ 6X^{-6}$	$- - - - + 201X^{-4}$	$1X^{-4}$
	&c.	&c.	&c.

Now the first Term of the moveable Paper, or Multiplier, being apply'd to the first Term of the Multiplicand, will give $5X^2 \times 3X = 15X^3$ for the first Term of the Product. Then the two first Terms of each being apply'd together, they will give $5X^2 \times 7X^0 + 2X \times 3X = 41X^2$ for the second Term of the Product. Then the three first Terms of each being apply'd together, they will give $5X^2 \times 5X^{-1} + 2X \times 7X^0 + 8X^0 \times 3X = 63X$ for the third Term of the Product. And so on. So that the Product required will be $15X^3 + 41X^2 + 63X + 97X^0 + 142X^{-1} + 133X^{-2} + 138X^{-3} + 201X^{-4}$, &c. Now this will be a Number in the Decimal Scale of Arithmetick, because $X = 10$. But in that Scale, when it is regular, the Coefficients must always be affirmative Integers, less than the Root 10; and therefore to reduce these to such, set them orderly under one another, as is done here, and beginning at the lowest, collect them as they stand, by adding up each Column. The reason of which is this. Because $201X^{-4} = 20X^{-3} + 1X^{-4}$, we must set down $1X^{-4}$, and add $20X^{-3}$ to the line above. Then because $20X^{-3} + 138X^{-3} = 158X^{-3} = 15X^{-2} + 8X^{-3}$, we must set down $8X^{-3}$, and add $15X^{-2}$ to the line above. Then because $15X^{-2} + 133X^{-2} = 148X^{-2} = 14X^{-1} + 8X^{-2}$, we must set down $8X^{-2}$, and add $14X^{-1}$ to the line above. And so we must proceed through the whole Number. So that at last we shall find the Product to be $1X^4 + 9X^3 + 8X^2 + 4X + 2X^0 + 6X^{-1} + 8X^{-2} + 8X^{-3}$, &c. Or by suppressing X , or 10, and leaving it to be supply'd by the Imagination, the Product required will be 19842,688, &c.

When one infinite Series is to be divided by another, write down the Terms of the Dividend, with their proper Signs, in a descending order, so that the Terms may be at equal distances, and just under

der one another. This is your fixt or right-hand Paper. Prepare another Paper, at the right-hand Edge of which write down the Terms of the Divisor in an ascending order, with all their Signs changed except the first, so that the Terms may be at the same equal distances as before, and just over one another. This will be your moveable or left-hand Paper. Apply your moveable Paper to your fixt Paper, so that the first Term of the Divisor may be over-against the first Term of the Dividend. Divide the first Term of the Dividend by the first Term of the Divisor, and set down the Quotient over-against them to the right-hand, for the first Term of the Quotient required. Move your moveable Paper a step lower, so that two of the first Terms of the Divisor may be over-against two of the first Terms of the Dividend. Collect the second Term of the Dividend, together with the Product of the first Term of the Quotient now found, multiply'd by the Terms over-against it in the left-hand Paper; these divided by the first Term of the Divisor will be the second Term of the Quotient required. Move your moveable Paper a step lower, so that three of the first Terms of the Divisor may stand over-against three of the first Terms of the Dividend. Collect the third Term of the Dividend, together with the two Products of the two first Terms of the Quotient now found, each being multiply'd into the Term over-against it, in the left-hand Paper. These divided by the first Term of the Divisor will be the third Term of the Quotient required. Move your moveable Paper a step lower, so that four of the first Terms of the Divisor may stand over-against four of the first Terms of the Dividend. Collect the fourth Term of the Dividend, together with the three Products of the three first Terms of the Quotient now found, each being multiply'd by the Term over-against it in the left-hand Paper. These divided by the first Term of the Divisor will be the fourth Term of the Quotient required. And so on to find the fifth, and the succeeding Terms.

For an Example let it be propos'd to divide the infinite Series $a^2 + \frac{1}{8}ax + \frac{1}{30}x^2 + \frac{121x^3}{1260a} + \frac{281x^4}{1250a^2}, \&c.$ by the Series $a + \frac{1}{2}x + \frac{x^2}{3a} + \frac{x^3}{4a^2} + \frac{x^4}{5a^3}, \&c.$ These being disposed as is prescribed, will stand as here follows.

Divisor,

Divisor,	Dividend						Quotient
$\mathcal{E}c.$	a^2	a	-----				a
$-\frac{x^4}{5a^3}$	$+\frac{1}{6}ax$	$+\frac{1}{6}ax$	$-\frac{1}{2}ax$	$=$	$-\frac{1}{3}ax$	-----	$-\frac{1}{3}x$
$-\frac{x^3}{4a^2}$	$+\frac{1}{3}x^2$	$+\frac{1}{3}x^2$	$+\frac{1}{6}x^2$	$-\frac{1}{3}x^2$	$=$	$+\frac{1}{5}x^2$	$+\frac{x^2}{5a}$
$-\frac{x^2}{3a}$	$+\frac{121x^3}{1260a}$	$+\frac{121x^3}{1260a}$	$+\frac{x^3}{10a}$	$+\frac{x^3}{9a}$	$-\frac{x^3}{4a}$	$=$	$-\frac{x^3}{7a^2}$
$-\frac{1}{2}x$	$+\frac{281x^4}{1260a^2}$	$+\frac{281x^4}{1260a^2}$	$+\frac{x^4}{14a^2}$	$-\frac{x^4}{15a^2}$	$+\frac{x^4}{12a^2}$	$-\frac{x^4}{5a^2}$	$+\frac{x^4}{9a^3}$
a	$\mathcal{E}c.$		$\mathcal{E}c.$				$\mathcal{E}c.$

Here if we apply the first Term of the Divisor a , to the first Term of the Dividend a^2 , by Division we shall have a for the first Term of the Quotient. Then applying the two first Terms of the Divisor to the two first Terms of the Dividend, we shall have $\frac{1}{6}ax$ to be collected with the Product $a \times -\frac{1}{2}x$, or $-\frac{1}{2}ax$, which will make $-\frac{1}{3}ax$; and this divided by a , the first Term of the Divisor, will give $-\frac{1}{3}x$ for the second Term of the Quotient. And so of the other Terms; and in like manner for all other Examples.

When an infinite Series is to be raised to any Power, or when any Root of it is to be extracted, it may be perform'd in all cases by a like Artifice. Prepare your fixt or right-hand Paper, by writing down the natural Numbers 0, 1, 2, 3, 4, &c. just under one another at equal distances, reserving places to the right-hand for the several Terms of the Power or Root, as they shall be found. The first Term of which Series may be immediately known from the first Term of the given Series, and from the given Index of the Power or Root, whether that Index be an Integer or a Fraction, affirmative or negative; and that Term therefore may be set down in its place, over-against the first Number 0. Prepare your moveable or left-hand Paper, by writing down, towards the edge of the Paper at the right-hand, all the Terms of the given Series, except the first, over one another in order, at the same distances as the Numbers in the other Paper. After which, nearer the edge of the Paper, write just over one another, first the Index of the Power or Root to be found, then its double, then its triple, and so the rest of its multiples, with the negative Sign after each, as far as the Terms of the Series extend. And also the first Term of the given Series may be wrote below. Thus will the moveable Paper be prepared. These multiples, together with the following negative Signs, and the Numbers

0, 1, 2, 3, 4, &c. on the other Paper, when they meet together, will compleat the numeral Coefficients. Apply therefore the second Term of the moveable Paper to the uppermost Term of the fixt Paper, and the Product made by the continual Multiplication of the three Factors that stand in a line over-against one another, [which are the second Term of the given Series, the numeral Coefficient, (here the given Index,) and the first Term of the Series already found,] divided by the first Term of the given Series, will be the second Term of the Series required, which is to be set down in its place over-against 1. Move the moveable Paper a step lower, and the two Products made by the multiplication of the Factors that stand over-against one another, (in which, and elsewhere, care must be had to take the numeral Coefficients compleat,) divided by twice the first Term of the given Series, will be the third Term of the Series required, which is to be set down in its place over-against 2. Move the moveable Paper a step lower, and the three Products made by the multiplication of the Factors that stand over-against one another, divided by thrice the first Term of the given Series, will be the fourth Term of the Series required. And so you may proceed to find the next, and the subsequent Terms.

It may not be amiss to give one general Example of this Reduction, which will comprehend all particular Cases. If the Series $ax + bz^2 + cz^3 + dz^4$, &c. be given, of which we are to find any Power, or to extract any Root; let the Index of this Power or Root be m . Then prepare the moveable or left-hand Paper as you see below, where the Terms of the given Series are set over one another in order, at the edge of the Paper, and at equal distances. Also after every Term is put a full point, as a Mark of Multiplication, and after every one, (except the first or lowest) are put the several Multiples of the Index, as $m, 2m, 3m, 4m$, &c. with the negative Sign — after them. Likewise a *vinculum* may be understood to be placed over them, to connect them with the other parts of the numeral Coefficients, which are on the other Paper, and which make them compleat. Also the first Term of the given Series is separated from the rest by a line, to denote its being a Divisor, or the Denominator of a Fraction. And thus is the moveable Paper prepared.

To prepare the fixt or right-hand Paper, write down the natural Numbers 0, 1, 2, 3, 4, &c. under one another, at the same equal distances as the Terms in the other Paper, with a Point after them as a Mark of Multiplication; and over-against the first Term o
write

write $a^m z^m$ for the first Term of the Series required. The rest of the Terms are to be wrote down orderly under this, as they shall be found, which will be in this manner. To the first Term o in the fixt Paper apply the second Term of the moveable Paper, and they will then exhibit this Fraction $\frac{bz^2. \overline{m-0. a^m z^m}}{az. 1}$, which being reduced to this $ma^{m-1}bz^{m+1}$, must be set down in its place, for the second Term of the Series required. Move the moveable Paper a step lower, and you will have this Fraction exhibited $\frac{+cz^3. \overline{2m-0. a^m z^m}}{+bz^2. \overline{m-1. ma^{m-1}bz^{m+1}}}$

which being reduced will become $ma^{m-1}c + m \times \frac{m-1}{2} a^{m-2}b^2 \times z^{m+2}$, to be put down for the third Term of the Series required. Bring down the moveable Paper a step lower, and you will have the Fraction $\frac{+dz^4. \overline{3m-0. a^m z^m}}{+cz^3. \overline{2m-1. ma^{m-1}bz^{m+1}}}$
 $\frac{+bz^2. \overline{m-2. ma^{m-1}c + m \times \frac{m-1}{2} a^{m-2}b^2 \times z^{m+2}}}{az. 3}$

which reduc'd will be $ma^{m-1}d + m \times \frac{m-1}{2} a^{m-2}bc + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3 \times z^{m+3}$, for the fourth Term of the Series required. And in the same manner are all the rest of the Terms to be found.

Moveable Paper, &c.	Fixt Paper
$o. a^m z^m$	$o. a^m z^m$
$+ dz^4. \overline{3m-}$	$1. ma^{m-1}bz^{m+1}$
$+ cz^3. \overline{2m-}$	$2. m \times \frac{m-1}{2} a^{m-2}b^2 + ma^{m-1}c \times z^{m+2}$
$+ bz^2. \overline{m-}$	$3. m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3 + m \times \frac{m-1}{2} a^{m-2}bc + ma^{m-1}d \times z^{m+3}$
$az.$	$\&c. \quad \&c.$

N. B. This Operation will produce Mr. De Moivre's Theorem mentioned before, the Investigation of which may be seen in the place there quoted, and shall be exhibited here in due time and place. And this therefore will sufficiently prove the truth of the present Process. In particular Examples this Method will be found very easy and practicable.

But now to shew something of the use of these Theorems, and at the same time to prepare the way for the Solution of Affected and Fluxional Equations; we will here make a kind of retrospect, and resume our Author's Examples of simple Extractions, beginning with Division itself, which we shall perform after a different and an easier manner.

Thus to divide aa by $b+x$, or to resolve the Fraction $\frac{aa}{b+x}$ into a Series of simple Terms; make $\frac{aa}{b+x} = y$, or $by + xy = aa$. Now to find the quantity y dispose the Terms of this Equation after this manner $\left. \begin{matrix} by \\ +xy \end{matrix} \right\} = a^2$, and proceed in the Resolution as you see is done here.

$$\begin{aligned} by \left\{ \right. &= a^2 - \frac{a^2x}{b} + \frac{a^2x^2}{b^2} - \frac{a^2x^3}{b^3} + \frac{a^2x^4}{b^4}, \&c. \\ +xy \left\{ \right. &- - - + \frac{a^2x}{b} - \frac{a^2x^2}{b^2} + \frac{a^2x^3}{b^3} - \frac{a^2x^4}{b^4}, \&c. \\ y &= \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} + \frac{a^2x^4}{b^5}, \&c. \end{aligned}$$

Here by the disposition of the Terms a^2 is made the first Term of the Series belonging (or equivalent) to by , and therefore dividing by b , $\frac{a^2}{b}$ will be the first Term of the Series equivalent to y , as is set down below. Then will $+\frac{a^2x}{b}$ be the first Term of the Series $+xy$, which is therefore set down over-against it; as also it is set down over-against by , but with a contrary Sign, to be the second Term of that Series. Then will $-\frac{a^2x^2}{b^2}$ be the second Term of y , to be set down in its place, which will give $-\frac{a^2x^2}{b^2}$ for the second Term of $+xy$; and this with a contrary Sign must be set down for the third Term of by . Then will $+\frac{a^2x^2}{b^3}$ be the third Term of y , and therefore $+\frac{a^2x^3}{b^3}$ will be the third Term of $+xy$, which with a contrary Sign must be made the fourth Term of by , and therefore $-\frac{a^2x^3}{b^4}$ will be the fourth Term of y . And so on for ever.

Now the *Rationale* of this Process, and of all that will here follow of the same kind, may be manifest from these Considerations. The unknown Terms of the Equation, or those wherein y is found, are (by the *Hypothesis*) equal to the known Term aa . And each of those

those unknown Terms is resolved into its equivalent Series, the Aggregate of which must still be equal to the same known Term aa ; (or perhaps Terms.) Therefore all the subsidiary and adventitious Terms, which are introduced into the Equation to assist the Solution, (or the Supplemental Terms,) must mutually destroy one another.

Or we may resolve the same Equation in the following manner:

$$\begin{aligned} by \} & \dots + \frac{ba^2}{x} - \frac{b^2a^2}{x^2} + \frac{b^3a^2}{x^3}, \&c. \\ +xy \} & = a^2 - \frac{ba^2}{x} + \frac{b^2a^2}{x^2} - \frac{b^3a^2}{x^3}, \&c. \\ y & = \frac{a^2}{x} - \frac{ba^2}{x^2} + \frac{b^2a^2}{x^3} - \frac{b^3a^2}{x^4}, \&c. \end{aligned}$$

Here a^2 is made the first Term of $+xy$, and therefore $\frac{a^2}{x}$ must be put down for the first Term of y . This will give $+\frac{ba^2}{x}$ for the first Term of by , which with a contrary Sign must be the second Term of $+xy$, and therefore $-\frac{ba^2}{x^2}$ must be put down for the second Term of y . Then will $-\frac{b^2a^2}{x^2}$ be the second Term of by , which with a contrary Sign will be the third Term of $+xy$, and therefore $+\frac{b^2a^2}{x^3}$ will be the third Term of y . And so on. Therefore the Fraction proposed is resolved into the same two Series as were found above.

If the Fraction $\frac{1}{1+x^2}$ were given to be resolved, make $\frac{1}{1+x^2} = y$, or $y + x^2y = 1$, the Resolution of which Equation is little more than writing down the Terms, in the manner following:

$$\begin{aligned} y \} & = 1 - x^2 + x^4 - x^6 + x^8, \&c. & y \} & \dots + x^{-2} - x^{-4} + x^{-6} - x^{-8}, \&c. \\ +x^2y \} & \dots + x^2 - x^4 + x^6, \&c. & +x^2y \} & = 1 - x^{-2} + x^{-4} - x^{-6}, \&c. \end{aligned}$$

Here in the first Paradigm, as 1 is made the first Term of y , so will x^2 be the first Term of x^2y , and therefore $-x^2$ will be the second Term of y , and therefore $-x^4$ will be the second Term of x^2y , and therefore $+x^4$ will be third Term of y ; &c. Also in the second Paradigm, as 1 is made the first Term of x^2y , so will $+x^{-2}$ be the first Term of y , and therefore $-x^{-2}$ will be the second Term of x^2y , or $-x^{-4}$ will be the second Term of y ; &c.

To resolve the compound Fraction $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x}$ into simple Terms, make $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = y$, or $2x^{\frac{1}{2}} - x^{\frac{3}{2}} = y + x^{\frac{1}{2}}y - 3xy$; which Equation may be thus resolved:

$$\begin{array}{r}
 = 2x^{\frac{1}{2}} \quad * \quad - x^{\frac{3}{2}} \\
 \hline
 y \left\{ \begin{array}{l} - - - 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}} - 73x^3, \&c. \\
 + x^{\frac{1}{2}}y \left\{ \begin{array}{l} - - - - - + 2x - 2x^{\frac{3}{2}} + 7x^2 - 13x^{\frac{5}{2}} + 34x^3, \&c. \\
 - 3xy \left\{ \begin{array}{l} - - - - - - - - - - - 6x^{\frac{3}{2}} + 6x^2 - 21x^{\frac{5}{2}} + 39x^3, \&c.
 \end{array}
 \end{array}
 \right.
 \end{array}
 \right.
 \end{array}$$

Place the Terms of the Equation, in which the unknown quantity y is found, in a regular descending order, and the known Terms above, as you see is done here. Then bring down $2x^{\frac{1}{2}}$ to be the first Term of y , which will give $+2x$ for the first Term of the Series $+x^{\frac{1}{2}}y$, which must be wrote with a contrary Sign for the second Term of y . Then will the second Term of $+x^{\frac{1}{2}}y$ be $-2x^{\frac{3}{2}}$, and the first Term of the Series $-3xy$ will be $-6x^{\frac{3}{2}}$, which together make $-8x^{\frac{3}{2}}$. And this with a contrary Sign would have been wrote for the third Term of y , had not the Term $-x^{\frac{3}{2}}$ been above, which reduces it to $+7x^{\frac{3}{2}}$ for the third Term of y . Then will $+7x^2$ be the third Term of $+x^{\frac{1}{2}}y$, and $+6x^2$ will be the second Term of $-3xy$, which being collected with a contrary Sign, will make $-13x^2$ for the fourth Term of y ; and so on, as in the Paradigm.

If we would resolve this Fraction, or this Equation, so as to accommodate it to the other case of convergency, we may invert the Terms, and proceed thus:

$$\begin{array}{r}
 = -x^{\frac{3}{2}} \quad * \quad + 2x^{\frac{1}{2}} \\
 \hline
 - 3xy \left\{ \begin{array}{l} - - - - - x^{\frac{3}{2}} - \frac{1}{3}x + \frac{1}{9}x^{\frac{4}{2}} + \frac{1}{27}, \&c. \\
 + x^{\frac{1}{2}}y \left\{ \begin{array}{l} - - - - - + \frac{1}{3}x + \frac{1}{9}x^{\frac{3}{2}} - \frac{1}{27}, \&c. \\
 + y \left\{ \begin{array}{l} - - - - - - - - - - - + \frac{1}{3}x^{\frac{1}{2}} + \frac{1}{9}, \&c.
 \end{array}
 \end{array}
 \right.
 \end{array}
 \right.
 \end{array}$$

$$y = \frac{1}{3}x^{\frac{1}{2}} + \frac{1}{9} - \frac{1}{27}x^{-\frac{1}{2}} - \frac{1}{81}x^{-1}, \&c.$$

Bring down $-x^{\frac{3}{2}}$ to be the first Term of $-3xy$, whence $+\frac{1}{3}x^{\frac{1}{2}}$ will be the first Term of y , to be set down in its place. Then the first

first Term of $+x^{\frac{1}{2}}y$ will be $+\frac{x}{3}$, which with a contrary Sign will be the second Term of $-3xy$, and therefore $+\frac{x}{9}$ will be the second Term of y . Then the second Term of $+x^{\frac{1}{2}}y$ will be $+\frac{x^{\frac{1}{2}}}{9}$, and the first Term of y being $+\frac{x}{3}$, these two collected with a contrary Sign would have made $-\frac{4}{9}x^{\frac{1}{2}}$ for the third Term of $-3xy$, had not the Term $+2x^{\frac{1}{2}}$ been present above. Therefore uniting these, we shall have $+\frac{14}{9}x^{\frac{1}{2}}$ for the third Term of $-3xy$, which will give $-\frac{14}{27}x^{-\frac{1}{2}}$ for the third Term of y . Then will the third Term of $+x^{\frac{1}{2}}y$ be $-\frac{14}{27}$, and the second Term of y being $+\frac{x}{9}$, these two collected with a contrary Sign will make $+\frac{14}{27}$ for the fourth Term of $-3xy$, and therefore $-\frac{14}{81}x^{-1}$ will be the fourth Term of y ; and so on.

And thus much for Division; now to go on to the Author's pure or simple Extractions.

To find the Square-root of $aa + xx$, or to extract the Root y of this Equation $yy = aa + xx$; make $y = a + p$, then we shall have by Substitution $2ap + pp = xx$, of which affected Quadratick Equation we may thus extract the Root p . Dispose the Terms in this manner $2ap \} = xx$, the unknown Terms in a descending order on one side, and the known Term or Terms on the other side of the Equation, and proceed in the Extraction as is here directed.

$$\begin{array}{l} 2ap \} = x^2 - \frac{x^4}{4a^2} + \frac{x^6}{8a^4} - \frac{5x^8}{64a^6} + \frac{7x^{10}}{128a^8}, \&c. \\ + p^2 \} - - - + \frac{x^4}{4a^2} - \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8}, \&c. \\ p = \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9}, \&c. \end{array}$$

By this Disposition of the Terms, x^2 is made the first Term of the Series belonging to $2ap$; then we shall have $\frac{x^2}{2a}$ for the first Term of the Series p , as here set down underneath. Therefore $\frac{x^4}{4aa}$ will be the first Term of the Series p^2 , to be put down in its place over-against p^2 . Then, by what is observed before, it must be put down with a contrary Sign as the second Term of $2ap$, which will make the second Term of p to be $-\frac{x^4}{8a^3}$. Having there-fore

fore the two first Terms of $p = \frac{x^2}{2a} - \frac{x^4}{8a^3}$, we shall have, (by any of the foregoing Methods for finding the Square of an infinite Series,) the two first Terms of $p^2 = \frac{x^4}{4a^2} - \frac{x^6}{8a^4}$; which last Term must be wrote with a contrary Sign, as the third Term of $2ap$. Therefore the third Term of p is $\frac{x^6}{16a^5}$, and the third Term of p^2 (by the aforesaid Methods) will be $\frac{5x^8}{64a^6}$, which is to be wrote with a contrary Sign, as the fourth Term of $2ap$. Then the fourth Term of p will be $-\frac{5x^8}{128a^7}$, and therefore the fourth Term of p^2 is $-\frac{7x^{10}}{128a^8}$, which is to be wrote with a contrary Sign for the fifth Term of $2ap$. This will give $\frac{7x^{10}}{256a^9}$ for the fifth Term of p ; and so we may proceed in the Extraction as far as we please.

Or we may dispose the Terms of the Supplemental Equation thus:

$$\left. \begin{array}{l} 2ap \\ + p^2 \end{array} \right\} \begin{array}{l} - - - + 2ax - 2a^2 + \frac{a^3}{x} * - \frac{a^5}{4x^3}, \&c. \\ = x^2 - 2ax + 2a^2 - \frac{a^3}{x} * + \frac{a^5}{4x^3}, \&c. \end{array}$$

$$p = x - a + \frac{a^2}{2x} * - \frac{a^4}{8x^3} *, \&c.$$

Here x^2 is made the first Term of the Series p^2 , and therefore x , (or else $-x$,) will be the first Term of p . Then $2ax$ will be the first Term of $2ap$, and therefore $-2ax$ will be the second Term of p^2 . So that because $p^2 = x^2 - 2ax$, &c. by extracting the Square-root of this Series by any of the foregoing Methods, it will be found $p = x - a$, &c. or $-a$ will be the second Term of the Root p . Therefore the second Term of $2ap$ will be $-2a^2$, which must be wrote with a contrary Sign for the third Term of p^2 , and thence (by Extraction) the third Term of p will be $\frac{a^2}{2x}$. This will make the third Term of $2ap$ to be $\frac{a^3}{x}$, which makes the fourth Term of p^2 to be $-\frac{a^3}{x}$, and therefore (by Extraction) 0 will be the fourth Term of p . This makes the fourth Term of $2ap$ to be 0, as also of p^2 . Then $-\frac{a^4}{8x^3}$ will be the fifth Term of p . Then the fifth Term of

$2ap$ will be $-\frac{a^5}{4x^3}$, which will make the sixth Term of p^2 to be $\frac{a^5}{4x^3}$; and therefore 0 will be the sixth Term of p , &c.

Here the Terms will be alternately deficient; so that in the given Equation $yy = aa + xx$, the Root will be $y = a + x - a + \frac{a^2}{2x}$, &c. that is $y = x + \frac{a^2}{2x} - \frac{a^4}{8x^3} + \frac{a^6}{16x^5}$, &c. which is the same as if we should change the order of the Terms, or if we should change a into x , and x into a .

If we would extract the Square-root of $aa - xx$, or find the Root y of the Equation $yy = aa - xx$; make $y = a + p$, as before; then $2ap + p^2 = -x^2$, which may be resolved as in the following Paradigm:

$$\begin{array}{l} 2ap \left\{ \begin{array}{l} = -x^2 - \frac{x^4}{4a^2} - \frac{x^6}{8a^4} - \frac{5x^8}{64a^6} - \frac{7x^{10}}{128a^8}, \text{ \&c.} \\ + p^2 \left\{ \begin{array}{l} - - - - - + \frac{x^4}{4a^2} + \frac{x^6}{8a^4} + \frac{5x^8}{64a^6} + \frac{7x^{10}}{128a^8}, \text{ \&c.} \\ p = -\frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} - \frac{7x^{10}}{256a^9}, \text{ \&c.} \end{array} \right. \end{array} \right. \end{array}$$

Here if we should attempt to make $-x^2$ the first Term of $+p^2$, we should have $\sqrt{-x^2}$, or $x\sqrt{-1}$, for the first Term of p ; which being impossible, shews no Series can be form'd from that Supposition.

To find the Square-root of $x - xx$, or the Root y in this Equation $yy = x - xx$, make $y = x^{\frac{1}{2}} + p$, then $x + 2x^{\frac{1}{2}}p + p^2 = x - xx$, or $2x^{\frac{1}{2}}p + p^2 = -x^2$, which may be resolved after this manner:

$$\begin{array}{l} 2x^{\frac{1}{2}}p \left\{ \begin{array}{l} = -x^2 - \frac{1}{4}x^3 - \frac{1}{8}x^4, \text{ \&c.} \\ + p^2 \left\{ \begin{array}{l} - - - - - + \frac{1}{4}x^3 + \frac{1}{8}x^4, \text{ \&c.} \\ p = -\frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}, \text{ \&c.} \end{array} \right. \end{array} \right. \end{array}$$

The Terms being rightly disposed, make $-x^2$ the first Term of $2x^{\frac{1}{2}}p$; then will $-\frac{1}{2}x^{\frac{3}{2}}$ be the first Term of p . Therefore $+\frac{1}{4}x^3$ will be the first Term of p^2 , which is also to be wrote with a contrary Sign for the second Term of $2x^{\frac{1}{2}}p$, which will give $-\frac{1}{8}x^{\frac{5}{2}}$ for the second Term of p . Then (by squaring) the second Term of p^2 will be $\frac{1}{8}x^4$, which will give $-\frac{1}{8}x^4$ for the second Term of $2x^{\frac{1}{2}}p$,

$2x^{\frac{1}{2}}p$, and therefore $-\frac{1}{16}x^{\frac{7}{2}}$ for the third Term of p ; and so on. Therefore in this Equation it will be $y = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$, &c.

So to extract the Root y of this Equation $yy = aa + bx - xx$, make $y = a + p$, then $2ap + p^2 = bx - xx$, which may be thus resolved.

$$\left. \begin{array}{l} 2ap \\ + p^2 \end{array} \right\} = \begin{array}{l} bx - x^2 + \frac{bx^3}{2a^2}, \text{ \&c.} \\ - \frac{b^2x^2}{4a^2} + \frac{b^3x^3}{8a^4} \\ + \frac{b^2x^2}{4a^2} - \frac{bx^3}{2a^2}, \text{ \&c.} \\ - \frac{b^3x^3}{8a^4} \end{array}$$

$$p = \frac{bx}{2a} - \frac{x^2}{2a} + \frac{bx^3}{4a^3}, \text{ \&c.}$$

$$- \frac{b^2x^2}{8a^3} + \frac{b^3x^3}{16a^5}$$

Make bx the first Term of $2ap$; then will $\frac{bx}{2a}$ be the first Term of p . Therefore the first Term of p^2 will be $+\frac{b^2x^2}{4a^2}$, which is also to be wrote with a contrary Sign, so that the second Term of $2ap$ will be $-x^2 - \frac{b^2x^2}{4a^2}$, which will make the second Term of p to be $-\frac{x^2}{2a} - \frac{b^2x^2}{8a^3}$. Then by squaring, the second Term of p^2 will be $-\frac{bx^3}{2a^2} - \frac{b^3x^3}{8a^4}$, which must be wrote with a contrary Sign for the third Term of $2ap$. This will give the third Term of p as in the Example; and so on. Therefore the Square-root of the Quantity $a^2 + bx - xx$ will be $a + \frac{bx}{2a} - \frac{x^2}{2a} - \frac{b^2x^2}{8a^3} + \frac{bx^3}{4a^3} + \frac{b^3x^3}{16a^5}$, &c.

Also if we would extract the Square-root of $\frac{1+ax^2}{1-bx^2}$, we may extract the Roots of the Numerator, and likewise of the Denominator, and then divide one Series by the other, as before; but more directly thus. Make $\frac{1+ax^2}{1-bx^2} = yy$, or $1+ax^2 = yy - b^2x^2y^2$. Suppose $y = 1 + p$, then $ax^2 = 2p + p^2 - bx^2 - 2bx^2p - bx^2p^2$, which Supplemental Equation may be thus resolved.

$$\begin{array}{l}
 2p \quad \left\{ \begin{array}{l} = ax^2 + \frac{1}{2}abx^4 + \frac{3}{8}ab^2x^6, \&c. \quad p = \frac{1}{2}ax^2 + \frac{1}{4}abx^4 + \frac{3}{16}ab^2x^6, \&c. \\ + b \quad + \frac{3}{4}b^2 \quad + \frac{5}{8}b^3 \quad + \frac{1}{2}b \quad + \frac{3}{8}b^2 \quad + \frac{5}{16}b^3 \\ - \frac{1}{4}a^2 \quad - \frac{1}{8}a^2b \quad - \frac{1}{8}a^2 \quad - \frac{1}{16}a^2b \\ + \frac{1}{8}a^3 \quad + \frac{1}{16}a^3 \end{array} \right. \\
 -2bx^2p \quad \left\{ \begin{array}{l} - ab \quad - \frac{1}{2}ab^2, \&c. \\ - b^2 \quad - \frac{3}{4}b^3 \\ + \frac{1}{4}a^2b \end{array} \right. \\
 +p^2 \quad \left\{ \begin{array}{l} + \frac{1}{4}a^2 \quad + \frac{1}{8}a^2b, \&c. \\ + \frac{1}{2}ab \quad + \frac{5}{8}ab^2 \\ + \frac{1}{4}b^2 \quad - \frac{1}{8}a^3 \\ + \frac{3}{8}b^3 \end{array} \right. \\
 -bx^2p^2 \quad \left\{ \begin{array}{l} - \frac{1}{4}a^2b, \&c. \\ - \frac{1}{2}ab^2 \\ - \frac{1}{4}b^3 \end{array} \right.
 \end{array}$$

Make $ax^2 + bx^2$ the first Term of $2p$, then will $\frac{1}{2}ax^2 + \frac{1}{2}bx^2$ be the first Term of p . Therefore $-abx^4 - b^2x^4$ will be the first Term of $-2bx^2p$, and $\frac{1}{4}a^2x^4 + \frac{1}{2}abx^4 + \frac{1}{4}bx^4$ will be the first Term of p^2 . These being collected, and their Signs changed, must be made the second Term of $2p$, which will give $\frac{1}{4}abx^4 + \frac{3}{8}b^2x^4 - \frac{1}{8}a^2x^4$ for the second Term of p . Then the second Term of $-2bx^2p$ will be $-\frac{1}{2}ab^2x^6 - \frac{3}{4}b^3x^6 + \frac{1}{4}a^2bx^6$, and the second Term of p^2 (by squaring) will be found $\frac{1}{8}a^2bx^6 + \frac{5}{8}ab^2x^6 - \frac{1}{8}a^3x^6 + \frac{3}{8}b^3x^6$, and the first Term of $-bx^2p^2$ will be $-\frac{1}{4}a^2bx^6 - \frac{1}{2}ab^2x^6 - \frac{1}{4}b^3x^6$; which being collected and the Signs changed, will make the third Term of $2p$, half which will be the third Term of p ; and so on as far as you please.

And thus if we were to extract the Cube-root of $a^3 + x^3$, or the Root y of this Equation $y^3 = a^3 + x^3$; make $y = a + p$, then by Substitution $a^3 + 3a^2p + 3ap^2 + p^3 = a^3 + x^3$, or $3a^2p + 3ap^2 + p^3 = x^3$, which supplemental Equation may be thus resolved.

$$\begin{array}{l}
 3a^2p \quad \left\{ \begin{array}{l} = x^3 - \frac{x^6}{3a^3} + \frac{5x^9}{27a^6} - \frac{10x^{12}}{81a^9}, \&c. \\ + 3ap^2 \quad \left\{ \begin{array}{l} - \frac{x^6}{3a^3} - \frac{2x^9}{9a^6} + \frac{13x^{12}}{81a^9}, \&c. \\ + p^3 \quad \left\{ \begin{array}{l} - \frac{x^9}{27a^6} - \frac{x^{12}}{27a^9}, \&c. \end{array} \right. \end{array} \right. \\
 p = \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}, \&c.
 \end{array}$$

The Terms being dispos'd in order, the first Term of the Series $3a^2p$ will be x^3 , which will make the first Term of p to be $\frac{x^3}{3a^2}$. This will make the first Term of p^2 to be $\frac{x^6}{9a^4}$. And this will make the first Term of $3ap^2$ to be $\frac{x^6}{3a^3}$, which with a contrary Sign must be the second Term of $3a^2p$, and therefore the second Term of p will be $-\frac{x^6}{9a^3}$. Then (by squaring) the second Term of $3ap^2$ will be $-\frac{2x^9}{9a^6}$, and (by cubing) the first Term of p^3 will be $\frac{x^9}{27a^6}$. These being collected make $-\frac{5x^9}{27a^6}$, which with a contrary Sign must be the third Term of $3a^2p$, and therefore the third Term of p will be $+\frac{5x^9}{81a^8}$. Then by squaring, the third Term of $3ap^2$ will be $\frac{13x^{12}}{81a^9}$, and by cubing, the second Term of p^3 will be $-\frac{x^{12}}{27a^9}$, which being collected will make $\frac{10x^{12}}{81a^9}$; and therefore the fourth Term of $3a^2p$ will be $-\frac{10x^{12}}{81a^9}$, and the fourth Term of p will be $-\frac{10x^{12}}{243a^{11}}$. And so on.

And thus may the Roots of all pure Equations be extracted, but in a more direct and simple manner by the foregoing Theorems. All that is here intended, is, to prepare the way for the Resolution of affected Equations, both in Numbers and Species, as also of Fluxional Equations, in which this Method will be found to be of very extensive use. And first we shall proceed with our Author to the Solution of numerical affected Equations.

SECT. III. *The Resolution of Numeral Affected Equations.*

19. **N**OW as to the Resolution of affected Equations, and first in Numbers; our Author very justly complains, that before his time the *exegeſis numeroſa*, or the Doctrine of the Solution of affected Equations in Numbers, was very intricate, defective, and inartificial. What had been done by *Vieta*, *Harriot*, and *Oughtred* in this matter, tho' very laudable Attempts for the time, yet however was extremely perplex'd and operose. So that he had good reason to reject their Methods, especially as he has substituted a much better in their room. They affected too great accuracy in pursuing exact

exact Roots, which led them into tedious perplexities; but he knew very well, that legitimate Approximations would proceed much more regularly and expeditiously, and would answer the same intention much better.

20, 21, 22. His Method may be easily apprehended from this one Instance, as it is contain'd in his Diagram, and the Explanation of it. Yet for farther Illustration I shall venture to give a short *rationale* of it. When a Numeral Equation is propos'd to be resolved, he takes as near an Approximation to the Root as can be readily and conveniently obtain'd. And this may always be had, either by the known Method of Limits, or by a Linear or Mechanical Construction, or by a few easy trials and suppositions. If this be greater or less than the Root, the Excess or Defect, indifferently call'd the Supplement, may be represented by p , and the assumed Approximation, together with this Supplement, are to be substituted in the given Equation instead of the Root. By this means, (expunging what will be superfluous,) a Supplemental Equation will be form'd, whose Root is now p , which will consist of the Powers of the assumed Approximation orderly descending, involved with the Powers of the Supplement regularly ascending, on both which accounts the Terms will be continually decreasing, in a decuple ratio or faster, if the assumed Approximation be suppos'd to be at least ten times greater than the Supplement. Therefore to find a new Approximation, which shall nearly exhaust the Supplement p , it will be sufficient to retain only the two first Terms of this Equation, and to seek the Value of p from the resulting simple Equation. [Or sometimes the three first Terms may be retain'd, and the Value of p may be more accurately found from the resulting Quadratick Equation; &c.] This new Approximation, together with a new Supplement q , must be substituted instead of p in this last supplemental Equation, in order to form a second, whose Root will be q . And the same things may be observed of this second supplemental Equation as of the first; and its Root, or an Approximation to it, may be discover'd after the same manner. And thus the Root of the given Equation may be prosecuted as far as we please, by finding new supplemental Equations, the Root of every one of which will be a correction to the preceding Supplement.

So in the present Example $y^3 - 2y - 5 = 0$, 'tis easy to perceive, that $y = 2$ *ferè*; for $2 \times 2 \times 2 - 2 \times 2 = 4$, which should make 5. Therefore let p be the Supplement of the Root, and it will be $y = 2 + p$, and therefore by substitution $-1 + 10p + 6p^2 + p^3 = 0$. As p is here suppos'd to be much less than the Approximation 2,

by this substitution an Equation will be form'd, in which the Terms will gradually decrease, and so much the faster, *cæteris paribus*, as 2 is greater than p . So taking the two first Terms, $-1 + 10p = 0$, *ferè*, or $p = \frac{1}{10}$ *ferè*; or assuming a second Supplement q , 'tis $p = \frac{1}{10} + q$ accurately. This being substituted for p in the last Equation, it becomes $0,61 + 11,23q + 6,3q^2 + q^3 = 0$, which is a new Supplemental Equation, in which all the Terms are farther deprefs'd, and in which the Supplement q will be much less than the former Supplement p . Therefore it is $0,61 + 11,23q = 0$, *ferè*, or $q = -\frac{0,61}{11,23}$ *ferè*, or $q = -0,0054 + r$ *accuratè*, by assuming r for the third Supplement. This being substituted will give $0,00054155 + 11,162r, \&c. = 0$, and therefore $r = -\frac{0,00054155}{11,162} = -0,00004852, \&c.$ So that at last $y = 2 + p = \&c.$ or $y = 2,09455148, \&c.$

And thus our Author's Method proceeds, for finding the Roots of affected Equations in Numbers. Long after this was wrote, Mr. *Raphson* publish'd his *Analysis Æquationum universalis*, containing a Method for the Solution of Numeral Equations, not very much different from this of our Author, as may appear by the following Comparison.

To find the Root of the Equation $y^3 - 2y = 5$, Mr. *Raphson* would proceed thus. His first Approximation he calls g , which he takes as near the true Root as he can, and makes the Supplement x , so that he has $y = g + x$. Then by Substitution $g^3 + 3g^2x + 3gx^2 + x^3 = 5$,
 $-2g - 2$

or if $g = 2$, 'tis $10x + 6x^2 + x^3 = 1$, to determine the Supplement x . This being supposed small, its Powers may be rejected, and therefore $10x = 1$, or $x = 0,1$ nearly. This added to g or 2, makes a new $g = 2,1$, and x being still the Supplement, 'tis $y = 2,1 + x$, which being substituted in the original Equation $y^3 - 2y = 5$, produces $11,23x + 6,3x^2 + x^3 = -0,61$, to determine the new Supplement x . He rejects the Powers of x , and thence derives

$x = \frac{-0,061}{11,23} = -0,0054$, and consequently $y = 2,0946$, which not being exact, because the Powers of x were rejected, he makes the Supplement again to be x , so that $y = 2,0946 + x$, which being substituted in the Original Equation, gives $11,162x + \&c. = -0,00054155$. Therefore to find the third Supplement x , he has
 $x = \frac{-0,00054155}{11,162} = -0,00004852$, so that $y = 2,0946 + x = 2,09455148, \&c.$ and so on.

By

By this Process we may see how nearly these two Methods agree, and wherein they differ. For the difference is only this, that our Author constantly prosecutes the Residual or Supplemental Equations, to find the first, second, third, &c. Supplements to the Root: But Mr. Raphson continually corrects the Root itself from the same supplemental Equations, which are formed by substituting the corrected Roots in the Original Equation. And the Rate of Convergency will be the same in both.

In imitation of these Methods, we may thus prosecute this Inquiry after a very general manner. Let the given Equation to be resolved be in this form $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3}, \&c. = 0$, in which suppose P to be any near Approximation to the Root y , and the little Supplement to be p . Then is $y = P + p$. Now from what is shewn before, concerning the raising of Powers and extracting Roots, it will follow that $y^m = P + p \mid^m = P^m + mP^{m-1}p, \&c.$ or that these will be the two first Terms of y^m ; and all the rest, being multiply'd into the Powers of p , may be rejected. And for the same reason $y^{m-1} = P^{m-1} + m-1P^{m-2}p, \&c.$ $y^{m-2} = P^{m-2} + m-2P^{m-3}p, \&c.$ and so of all the rest. Therefore these being substituted into the Equation, it will be

$$\left. \begin{aligned} &aP^m + maP^{m-1}p, \&c. \\ &+ bP^{m-1} + m-1bP^{m-2}p, \&c. \\ &+ cP^{m-2} + m-2cP^{m-3}p, \&c. \\ &+ dP^{m-3} + m-3dP^{m-4}p, \&c. \\ &\&c. \end{aligned} \right\} = 0; \text{ Or dividing by } P^m, \\ a + bP^{-1} + cP^{-2} + dP^{-3}, \&c. + maP^{-1}p + m-1bP^{-2}p + m-2cP^{-3}p \\ + m-3dP^{-4}p, \&c. = 0. \text{ From whence taking the Value of } p, \\ \text{we shall have } p = -\frac{a + bP^{-1} + cP^{-2} + dP^{-3}, \&c.}{maP^{-1} + m-1bP^{-2} + m-2cP^{-3} + m-3dP^{-4}, \&c.}, \text{ and} \\ \text{consequently } y = (P + p = P - \frac{a + bP^{-1} + cP^{-2} + dP^{-3}, \&c.}{maP^{-1} + m-1bP^{-2} + m-2cP^{-3} + m-3dP^{-4}, \&c.}) \\ \frac{m-1a + m-2bP^{-1} + m-3cP^{-2} + m-4dP^{-3}, \&c.}{maP^{-1} + m-1bP^{-2} + m-2cP^{-3} + m-3dP^{-4}, \&c.} =)$$

To reduce this to a more commodious form, make $P = \frac{A}{B}$, whence $P^{-1} = A^{-1}B$, $P^{-2} = A^{-2}B^2$, &c. which being substituted, and also multiplying the Numerator and Denominator by A^m , it will be $y = \frac{m-1aA^m + m-2bA^{m-1}B + m-3cA^{m-2}B^2 + m-4dA^{m-3}B^3, \&c.}{maA^{m-1}B + m-1bA^{m-2}B^2 + m-2cA^{m-3}B^3 + m-3dA^{m-4}B^4, \&c.}$ which will be a nearer Approach to the Root y , than $\frac{A}{B}$, or P , and so much the

the nearer as $\frac{A}{B}$ is near the Root. And hence we may derive a very convenient and general Theorem for the Extraction of the Roots of Numeral Equations, whether pure or affected, which will be this.

Let the general Equation $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3}, \&c. = 0$ be proposed to be solved; if the Fraction $\frac{A}{B}$ be assumed as near the Root y as conveniently may be, the Fraction $\frac{m-1aA^m + m-2bA^{m-1}B + m-3cA^{m-2}B^2 + m-4dA^{m-3}B^3, \&c.}{mA^{m-1}B + m-1bA^{m-2}B^2 + m-2cA^{m-3}B^3 + m-3dA^{m-4}B^4, \&c.}$ will be still a nearer Approximation to the Root. And this Fraction, when computed, may be used instead of the Fraction $\frac{A}{B}$, by which means a nearer Approximation may again be had; and so on, till we approach as near the true Root as we please.

This general Theorem may be conveniently resolved into as many particular Theorems as we please. Thus in the Quadratick Equation $y^2 + by = c$, it will be $y = \frac{A^2 + cB^2}{2A + bB \times B}$, *ferè*. In the Cubick Equation $y^3 + by^2 + cy = d$, it will be $y = \frac{2A + bB \times A^2 + dB^3}{3A^2 + 2bAB + cB^2 \times B}$, *ferè*. In the Biquadratick Equation $y^4 + by^3 + cy^2 + dy = e$, it will be $y = \frac{3A^2 + 2bAB + cB^2 \times A^2 + eB^4}{4A^3 + 3bA^2B + 2cAB^2 + dB^3 \times B}$, *ferè*. And the like of higher Equations.

For an Example of the Solution of a Quadratick Equation, let it be proposed to extract the Square-root of 12, or let us find the value of y in this Equation $y^2 = 12$. Then by comparing with the general formula, we shall have $b = 0$, and $c = 12$. And taking 3 for the first approach to the Root, or making $\frac{A}{B} = \frac{3}{1}$, that is, $A = 3$ and $B = 1$, we shall have by Substitution $y = \frac{9+12}{6} = \frac{7}{2}$, for a nearer Approximation. Again, making $A = 7$ and $B = 2$, we shall have $y = \frac{49+48}{14 \times 2} = \frac{97}{28}$ for a nearer Approximation. Again, making $A = 97$ and $B = 28$, we shall have $y = \frac{97^2 + 12 \times 28^2}{194 \times 28} = \frac{18817}{5432}$ for a nearer Approximation. Again, making $A = 18817$ and $B = 5432$, we shall have $y = \frac{18817^2 + 12 \times 5432^2}{37634 \times 5432} = \frac{708158977}{204427888}$ for a nearer Approximation. And if we go on in the same method, we may find as near an Approximation to the Root as we please.

This

This Approximation will be exhibited in a vulgar Fraction, which, if it be always kept to its lowest Terms, will give the Root of the Equation in the shortest and simplest manner. That is, it will always be nearer the true Root than any other Fraction whatever, whose Numerator and Denominator are not much larger Numbers than its own. If by Division we reduce this last Fraction to a Decimal, we shall have 3,46410161513775459 for the Square-root of 12, which exceeds the truth by less than an Unit in the last place.

For an Example of a Cubick Equation, we will take that of our Author $y^3 - 2y = 5$, and therefore by Comparison $b = 0$, $c = -2$, and $d = 5$. And taking 2 for the first Approach to the Root, or making $\frac{A}{B} = \frac{2}{1}$, that is, $A = 2$ and $B = 1$, we shall have by Substitution $y = \frac{16+5}{12-2} = \frac{21}{10}$ for a nearer Approach to the Root. Again, make $A = 21$ and $B = 10$, and then we shall have $y = \frac{9261+2500}{6615-1000} = \frac{11761}{5615}$ for a nearer Approximation. Again, make $A = 11761$ and $B = 5615$, and we shall have $y = \frac{2 \times 11761^3 + 5 \times 5615^3}{3 \times 11761^2 \times 5615 - 2 \times 5615^3} = \frac{4138744325037}{1975957316495}$ for a nearer Approximation. And so we might proceed to find as near an Approximation as we think fit. And when we have computed the Root near enough in a Vulgar Fraction, we may then (if we please) reduce it to a Decimal by Division. Thus in the present Example we shall have $y = 2,094551481701$, &c. And after the same manner we may find the Roots of all other numeral affected Equations, of whatever degree they may be.

SECT. IV. *The Resolution of Specious Equations by infinite Series; and first for determining the forms of the Series, and their initial Approximations.*

23, 24. **F**ROM the Resolution of numeral affected Equations, our Author proceeds to find the Roots of Literal, Specious, or Algebraical Equations also, which Roots are to be exhibited by an infinite converging Series, consisting of simple Terms. Or they are to be express'd by Numbers belonging to a general Arithmetical Scale, as has been explain'd before, of which the Root is denoted by x or z . The assigning or chusing this Root is what he means here, by distinguishing one of the literal Coefficients from the rest, if there are several. And this is done by ordering or disposing the

the Terms of the given Equation, according to the Dimensions of that Letter or Coefficient. It is therefore convenient to chuse such a Root of the Scale, (when choice is allow'd,) as that the Series may converge as fast as may be. If it be the least, or a Fraction less than Unity, its ascending Powers must be in the Numerators of the Terms. If it be the greatest quantity, then its ascending Powers must be in the Denominators, to make the Series duly converge. If it be very near a given quantity, then that quantity may be conveniently made the first Approximation, and that small difference, or Supplement, may be made the Root of the Scale, or the converging quantity. The Examples will make this plain.

25, 26. The Equation to be resolved, for conveniency-sake, should always be reduced to the simplest form it can be, before its Resolution be attempted; for this will always give the least trouble. But all the Reductions mention'd by the Author, and of which he gives us Examples, are not always necessary, tho' they may be often convenient. The Method is general, and will find the Roots of Equations involving fractional or negative Powers, as well as of other Equations, as will plainly appear hereafter.

27, 28. When a literal Equation is given to be resolved, in distinguishing or assigning a proper quantity, by which its Root is to converge, the Author before has made three cases or varieties; all which, for the sake of uniformity, he here reduces to one. For because the Series must necessarily converge, that quantity must be as small as possible, in respect of the other quantities, that its ascending Powers may continually diminish. If it be thought proper to chuse the greatest quantity, instead of that its Reciprocal must be introduced, which will bring it to the foregoing case. And if it approach near to a given quantity, then their small difference may be introduced into the Equation, which again will bring it to the first case. So that we need only pursue that case, because the Equation is always suppos'd to be reduced to it.

But before we can conveniently explain our Author's Rule, for finding the first Term of the Series in any Equation, we must consider the nature of those Numbers, or Expressions, to which these literal Equations are reduced, whose Roots are required; and in this Inquiry we shall be much assisted by what has been already discoursed of Arithmetical Scales. In affected Equations that were purely numerical, the Solution of which was just now taught, the several Powers of the Root were orderly disposed, according to a single or simple Arithmetical Scale, which proceeded only *in longum*, and was there sufficient

sufficient for their Solution. But we must enlarge our views in these literal affected Equations, in which are found, not only the Powers of the Root to be extracted, but also the Powers of the Root of the Scale, or of the converging quantity, by which the Series for the Root of the Equation is to be form'd; on account of each of which circumstances the Terms of the Equation are to be regularly disposed, and therefore are to constitute a double or combined Arithmetical Scale, which must proceed both ways, *in latum* as well as *in longum*, as it were in a Table. For the Powers of the Root to be extracted, suppose y , are to be disposed *in longum*, so as that their Indices may constitute an Arithmetical Progression, and the vacancies, if any, may be supply'd by the Mark *. Also the Indices of the Powers of the Root, by which the Series is to converge, suppose x , are to be disposed *in latum*, so as to constitute an Arithmetical Progression, and the vacancies may likewise be fill'd up by the same Mark *, when it shall be thought necessary. And both these together will make a combined or double Arithmetical Scale. Thus if the Equation $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + b^2x^4 = 0$, were given, to find the Root y , the Terms may be thus disposed :

	y^6	y^5	y^4	y^3	y^2	y^1	y^0
x^0	y^6	*	*	*	*	*	*
x^1	*	$-5xy^5$	*	*	*	*	*
x^2	*	*	*	*	$-7a^2x^2y^2$	*	*
x^3	*	*	$+\frac{x^3}{a}y^4$	*	*	*	$+6a^3x^3$
x^4	*	*	*	*	*	*	$+b^2x^4$

$= 0$

Also the Equation $y^5 - by^2 + 9bx^2 - x^3 = 0$ should be thus disposed, in order to its Solution:

y^5	*	*	$-by^2$	*	*
				*	*
				$+9bx^2$	*
				$-x^3$	*

$= 0$

And the Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$ thus:

y^3	*	$+a^2y$	$-2a^3$
		$+axy$	*
		*	*
		$-x^3$	*

$= 0$

And the Equation $x^2y^5 - 3c^4xy^2 - c^5x^2 + c^7 = 0$ thus:

*	*	*	*	*	$+c^7$
*	*	*	$-3c^4xy^2$	*	*
x^2y^5	*	*	*	$-c^5x^2$	*

$= 0$

And the like of all other Equations.

When the Terms of the Equation are thus regularly dispos'd, it is then ready for Solution; to which the following Speculation will be a farther preparation.

29. This ingenious contrivance of our Author, (which we may call Tabulating the Equation,) for finding the first Term of the Root, (which may indeed be extended to the finding all the Terms, or the form of the Series, or of all the Series that may be derived from the given Equation,) cannot be too much admired, or too carefully inquired into: The reason and foundation of which may be thus generally explain'd from the following Table, of which the Construction is thus.

$-2a+6b$	$-a+6b$	$+6b$	$a+6b$	$2a+6b$	$3a+6b$	$4a+6b$	$5a+6b$	$6a+6b$	$7a+6b$
$-2a+5b$	$-a+5b$	$+5b$	$a+5b$	$2a+5b$	$3a+5b$	$4a+5b$	$5a+5b$	$6a+5b$	$7a+5b$
$-2a+4b$	$-a+4b$	$+4b$	$a+4b$	$2a+4b$	$3a+4b$	$4a+4b$	$5a+4b$	$6a+4b$	$7a+4b$
$-2a+3b$	$-a+3b$	$+3b$	$a+3b$	$2a+3b$	$3a+3b$	$4a+3b$	$5a+3b$	$6a+3b$	$7a+3b$
$-2a+2b$	$-a+2b$	$+2b$	$a+2b$	$2a+2b$	$3a+2b$	$4a+2b$	$5a+2b$	$6a+2b$	$7a+2b$
$-2a+b$	$-a+b$	$+b$	$a+b$	$2a+b$	$3a+b$	$4a+b$	$5a+b$	$6a+b$	$7a+b$
$-2a$	$-a$	0	a	$2a$	$3a$	$4a$	$5a$	$6a$	$7a$
$-2a-b$	$-a-b$	$-b$	$a-b$	$2a-b$	$3a-b$	$4a-b$	$5a-b$	$6a-b$	$7a-b$
$-2a-2b$	$-a-2b$	$-2b$	$a-2b$	$2a-2b$	$3a-2b$	$4a-2b$	$5a-2b$	$6a-2b$	$7a-2b$
$-2a-3b$	$-a-3b$	$-3b$	$a-3b$	$2a-3b$	$3a-3b$	$4a-3b$	$5a-3b$	$6a-3b$	$7a-3b$

In a Plane draw any number of Lines, parallel and equidistant, and others at right Angles to them, so as to divide the whole Space, as far as is necessary, into little equal Parallelograms. Assume any one of these, in which write the Term 0 , and the Terms a , $2a$, $3a$, $4a$, &c. in the succeeding Parallelograms to the right hand, as also the Terms $-a$, $-2a$, $-3a$, &c. to the left hand. Over the Term 0 , in the same Column, write the Terms b , $2b$, $3b$, $4b$, &c. successively, and the Terms $-b$, $-2b$, $-3b$, &c. underneath. And these we may call primary Terms. Now to insert its proper Term in any other assign'd Parallelogram, add the two primary Terms together that stand over-against it each way, and write the Sum in the given Parallelogram. And thus all the Parallelograms being fill'd, as far as there is occasion every way, the whole Space will

will become a Table, which may be called a *combined Arithmetical Progression in plano*, composed of the two general Numbers a and b , of which these following will be the chief properties.

Any Row of Terms, parallel to the primary Series $0, a, 2a, 3a, \&c.$ will be an Arithmetical Progression, whose common Difference is a ; and it may be any such Progression at pleasure. Any Row or Column parallel to the primary Series $0, b, 2b, 3b, \&c.$ will be an Arithmetical Progression, whose common difference is b ; and it may be any such Progression. If a strait Ruler be laid on the Table, the Edge of which shall pass thro' the Centers of any two Parallelograms whatever; all the Terms of the Parallelograms, whose Centers shall at the same time touch the Edge of the Ruler, will constitute an Arithmetical Progression, whose common difference will consist of two parts, the first of which will be some Multiple of a , and the other a Multiple of b . If this Progression be suppos'd to proceed *inferiora versus*, or from the upper Term or Parallelogram towards the lower; each part of the common difference may be separately found, by subtracting the primary Term belonging to the lower, from the primary Term belonging to the upper Parallelogram. If this common difference, when found, be made equal to nothing, and thereby the Relation of a and b be determined; the Progression degenerates into a Rank of Equals, or (if you please) it becomes an Arithmetical Progression, whose common difference is infinitely little. In which case, if the Ruler be moved by a parallel motion, all the Terms of the Parallelograms, whose Centers shall at the same time be found to touch the Edge of the Ruler, shall be equal to each other. And if the motion of the Ruler be continued, such Terms as at equal distances from the first situation are successively found to touch the Ruler, shall form an Arithmetical Progression. Lastly, to come nearer to the case in hand, if any number of these Parallelograms be mark'd out and distinguish'd from the rest, or assign'd promiscuously and at pleasure, through whose Centers, as before, the Edge of the Ruler shall successively pass in its parallel motion, beginning from any two (or more) initial or external Parallelograms, whose Terms are made equal; an Arithmetical Progression may be found, which shall comprehend and take in all those promiscuous Terms, without any regard had to the Terms that are to be omitted. These are some of the properties of this Table, or of a combined Arithmetical Progression *in plano*, by which we may easily understand our Author's expedient, of Tabulating the given Equation, and may derive the necessary Consequences from it.

For when the Root y is to be extracted out of a given Equation, consisting of the Powers of y and x any how combined together promiscuously, with other known quantities, of which x is to be the Root of the Scale, (or Series,) as explain'd before; such a value of y is to be found, as when substituted in the Equation instead of y , the whole shall be destroy'd, and become equal to nothing. And first the initial Term of the Series, or the first Approximation, is to be found, which in all cases may be Analytically represented by Ax^m ; or we may always put $y = Ax^m$, &c. So that we shall have $y^2 = A^2x^{2m}$, &c. $y^3 = A^3x^{3m}$, &c. $y^4 = A^4x^{4m}$, &c. And so of other Powers or Roots. These when substituted in the Equation, and by that means compounded with the several Powers of x (or z) already found there, will form such a combined Arithmetical Progression *in plano* as is above described, or which may be reduced to such, by making $a = m$ and $b = 1$. These Terms therefore, according to the nature of the Equation, will be promiscuously dispersed in the Table; but the vacancies may always be conceived to be supply'd, and then it will have the properties before mention'd. That is, the Ruler being apply'd to two (or perhaps more) initial or external Terms, (for if they were not external, they could not be at the beginning of an Arithmetical Progression, as is necessarily required,) and those Terms being made equal, the general Index m will thereby be determined, and the general Coefficient A will also be known. If the external Terms made choice of are the lowest in the Table, which is the case our Author pursues, the Powers of x will proceed by increasing. But the highest may be chosen, and then a Series will be found, in which the Powers of x will proceed by decreasing. And there may be other cases of external Terms, each of which will commonly afford a Series. The initial Index being thus found, the other compound Indices belonging to the Equation will be known also, and an Arithmetical Progression may be found, in which they are all comprehended, and consequently the form of the Series will be known.

Or instead of Tabulating the Indices of the Equation, as above, it will be the same thing in effect, if we reduce the Terms themselves to the form of a combined Arithmetical Progression, as was shewn before. But then due care must be taken, that the Terms may be rightly placed at equal distances; otherwise the Ruler cannot be actually apply'd, to discover the Progressions of the Indices, as may be done in the Parallelogram.

For

For the sake of greater perspicuity, we will reduce our general Table, or combined Arithmetical Progression *in plano*, to the particular case, in which $a = m$ and $b = 1$; which will then appear thus:

$-2m+6$	$-m+6$	$+6$	$m+6$	$2m+6$	$3m+6$	$4m+6$	$5m+6$	$6m+6$	$7m+6$
$-2m+5$	$-m+5$	$+5$	$m+5$	$2m+5$	$3m+5$	$4m+5$	$5m+5$	$6m+5$	$7m+5$
$-2m+4$	$-m+4$	$+4$	$m+4$	$2m+4$	$3m+4$	$4m+4$	$5m+4$	$6m+4$	$7m+4$
$-2m+3$	$-m+3$	$+3$	$m+3$	$2m+3$	$3m+3$	$4m+3$	$5m+3$	$6m+3$	$7m+3$
$-2m+2$	$-m+2$	$+2$	$m+2$	$2m+2$	$3m+2$	$4m+2$	$5m+2$	$6m+2$	$7m+2$
$-2m+1$	$-m+1$	$+1$	$m+1$	$2m+1$	$3m+1$	$4m+1$	$5m+1$	$6m+1$	$7m+1$
$-2m$	$-m$	0	m	$2m$	$3m$	$4m$	$5m$	$6m$	$7m$
$-2m-1$	$-m-1$	-1	$m-1$	$2m-1$	$3m-1$	$4m-1$	$5m-1$	$6m-1$	$7m-1$
$-2m-2$	$-m-2$	-2	$m-2$	$2m-2$	$3m-2$	$4m-2$	$5m-2$	$6m-2$	$7m-2$
$-2m-3$	$-m-3$	-3	$m-3$	$2m-3$	$3m-3$	$4m-3$	$5m-3$	$6m-3$	$7m-3$

Now the chief properties of this Table, subservient to the present purpose, will be these. If any Parallelogram be selected, and another any how below it towards the right hand, and if their included Numbers be made equal, by determining the general Number m , which in this case will always be affirmative; also if the Edge of the Ruler be apply'd to the Centers of these two Parallelograms; all the Numbers of the other Parallelograms, whose Centers at the same time touch the Ruler, will likewise be equal to each other. Thus if the Parallelogram denoted by $m+4$ be selected, as also the Parallelogram $3m+2$; and if we make $m+4 = 3m+2$, we shall have $m = 1$. Also the Parallelograms $-m+6$, $m+4$, $3m+2$, $5m$, $7m-2$, &c. will at the same time be found to touch the Edge of the Ruler, every one of which will make 5, when $m = 1$.

And the same things will obtain if any Parallelogram be selected, and another any how below it towards the left-hand, if their included Numbers be made equal, by determining the general Number m , which in this case will be always negative. Thus if the Parallelogram denoted by $5m+4$ be selected, as also the Parallelogram $4m+2$; and if we make $5m+4 = 4m+2$, we shall have $m = -2$. Also the Parallelograms $6m+6$, $5m+4$, $4m+2$, $3m$, $2m-2$, &c. will

will be found at the same time to touch the Ruler, every one of which will make -6 , when $m = -2$.

The same things remaining as before, if from the first situation of the Ruler it shall move towards the right-hand by a parallel motion, it will continually arrive at greater and greater Numbers, which at equal distances will form an ascending Arithmetical Progression. Thus if the two first selected Parallelograms be $2m - 1 = 5m - 3$, whence $m = \frac{2}{3}$, the Numbers in all the corresponding Parallelograms will be $\frac{2}{3}$. Then if the Ruler moves towards the right-hand, into the parallel situation $3m + 1$, $6m - 1$, &c. these Numbers will each be 3. If it moves forwards to the same distance, it will arrive at $4m + 3$, $7m + 1$, &c. which will each be $5\frac{2}{3}$. If it moves forward again to the same distance, it will arrive at $5m + 5$, $8m + 3$, &c. which will each be $8\frac{2}{3}$. And so on. But the Numbers $\frac{2}{3}$, 3, $5\frac{2}{3}$, $8\frac{2}{3}$, &c. are in an Arithmetical Progression whose common difference is $2\frac{2}{3}$. And the like, *mutatis mutandis*, in other circumstances.

And hence it will follow *è contrà*, that if from the first situation of the Ruler, it moves towards the left-hand by a parallel motion, it will continually arrive at lesser and lesser Numbers, which at equal distances will form a decreasing Arithmetical Progression.

But in the other situation of the Ruler, in which it inclines downwards towards the left-hand, if it be moved towards the right-hand by a parallel motion, it will continually arrive at greater and greater Numbers, which at equal distances will form an increasing Arithmetical Progression. Thus if the two first selected Numbers or Parallelograms be $8m + 1 = 5m - 1$, whence $m = -\frac{2}{3}$, and the Numbers in all the corresponding Parallelograms will be $-4\frac{2}{3}$. If the Ruler moves upwards into the parallel situation $5m + 2$, $2m$, &c. these Numbers will each be $-1\frac{2}{3}$. If it move on at the same distance, it will arrive at $2m + 3$, $-m + 1$, &c. which will each be $1\frac{2}{3}$. If it move forward again to the same distance, it will arrive at $-m + 4$, $-4m + 2$, &c. which will each be $4\frac{2}{3}$. And so on. But the Numbers $-4\frac{2}{3}$, $-1\frac{2}{3}$, $1\frac{2}{3}$, $4\frac{2}{3}$, &c. or $-\frac{13}{3}$, $-\frac{4}{3}$, $\frac{5}{3}$, $\frac{14}{3}$, &c. are in an increasing Arithmetical Progression, whose common difference is $\frac{2}{3}$, or 3.

And hence it will follow also, if in this last situation of the Ruler it moves the contrary way, or towards the left-hand, it will continually arrive at lesser and lesser Numbers, which at equal distances will form a decreasing Arithmetical Progression.

Now if out of this Table we should take promiscuously any number of Parallelograms, in their proper places, with their respective Num-

Numbers included, neglecting all the rest; we should form some certain Figure, such as this, of which these would be the properties.

		$3m+5$		$5m+5$	
$m+3$			$4m+3$		$6m+3$
	$2m+1$			$5m+1$	

The Ruler being apply'd to any two (or perhaps more) of the Parallelograms which are in the Ambit or Perimeter of the Figure, that is, to two of the external Parallelograms, and their Numbers being made equal, by determining the general Number m ; if the Ruler passes over all the rest of the Parallelograms by a parallel motion, those Numbers which at the same time come to the Edge of the Ruler will be equal, and those that come to it successively will form an Arithmetical Progression, if the Terms should lie at equal distances; or at least they may be reduced to such, by supplying any Terms that may happen to be wanting.

Thus if the Ruler should be apply'd to the two uppermost and external Parallelograms, which include the Numbers $3m+5$ and $5m+5$, and if they be made equal, we shall have $m=0$, so that each of these Numbers will be 5. The next Numbers that the Ruler will arrive at will be $m+3$, $4m+3$, $6m+3$, of which each will be 3. The last are $2m+1$, $5m+1$, of which each is 1. So that here $m=0$, and the Numbers arising are 5, 3, 1, which form a decreasing Arithmetical Progression, the common difference of which is 2. And if there had been more Parallelograms, any how disposed, their Numbers would have been comprehended by this Arithmetical Progression, or at least it might have been interpolated with other Terms, so as to comprehend them all, however promiscuously and irregularly they might have been taken.

Thus secondly, if the Ruler be apply'd to the two external Parallelograms $5m+5$ and $6m+3$, and if these Numbers be made equal, we shall have $m=2$, and the Numbers themselves will be each 15. The three next Numbers which the Ruler will arrive at will

will be each 11, and the two last will be each 5. But the Numbers 15, 11, 5, will be comprehended in the decreasing Arithmetical Progression 15, 13, 11, 9, 7, 5, whose common difference is 2.

Thirdly, if the Ruler be apply'd to the two external Parallelograms $6m + 3$ and $5m + 1$, and if these Numbers be made equal, we shall have $m = -2$, and the Numbers will be each -9 . The two next Numbers that the Ruler will arrive at will be each -5 , the next will be -3 , the next -1 , and the last $+1$. All which will be comprehended in the ascending Arithmetical Progression $-9, -7, -5, -3, -1, +1$, whose common difference is 2.

Fourthly, if the Ruler be apply'd to the two lowest and external Parallelograms $2m + 1$ and $5m + 1$, and if they be made equal, we shall have again $m = 0$, so that each of these Numbers will be 1. The next three Numbers that the Ruler will approach to, will each be 3, and the last 5. But the Numbers 1, 3, 5, will be comprehended in an ascending Arithmetical Progression, whose common difference is 2.

Fifthly, if the Ruler be apply'd to the two external Parallelograms $m + 3$ and $2m + 1$, and if these Numbers be made equal, we shall have $m = 2$, and the Numbers themselves will be each 5. The three next Numbers that the Ruler will approach to will each be 11, and the two next will be each 15. But the Numbers 5, 11, 15, will be comprehended in the ascending Arithmetical Progression 5, 7, 9, 11, 13, 15, of which the common difference is 2.

Lastly, if the Ruler be apply'd to the two external Parallelograms $3m + 5$ and $m + 3$, and if these Numbers be made equal, we shall have $m = -1$, and the Numbers themselves will each be 2. The next Number to which the Ruler approaches will be 0, the two next are each -1 , the next -3 , the last -4 . All which Numbers will be found in the descending Arithmetical Progression 2, 1, 0, $-1, -2, -3, -4$, whose common difference is 1. And these six are all the possible cases of external Terms.

Now to find the Arithmetical Progression, in which all these resulting Terms shall be comprehended; find their differences, and the greatest common Divisor of those differences shall be the common difference of the Progression. Thus in the fifth case before, the resulting Numbers were 5, 11, 15, whose differences are 6, 4, and their greatest common Divisor is 2. Therefore 2 will be the common difference of the Arithmetical Progression, which will include all the resulting Numbers 5, 11, 15, without any superfluous Terms. But the application of all this will be best apprehended from the Examples that are to follow.

30. We have before given the form of this Equation, $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^3 + 6a^3x^3 + b^2x^4 = 0$, when the Terms are disposed according to a double or combined Arithmetical Scale, in order to its Solution. Or observing the same disposition of the Terms, they may be inserted in their respective Parallelograms, as the Table requires. Or rather, it may be sufficient to tabulate the several Indices of x only, when they are derived as follows. Let Ax^m represent the first Term of the Series to be form'd for y , as before, or let $y = Ax^m$, &c. Then by substituting this for y in the given Equation, we shall have $A^6x^{6m} - 5A^5x^{5m+1} + \frac{A^4}{a}x^{4m+3} - 7a^2A^3x^{3m+2} + 6a^3x^3 + b^2x^4$, &c. $= 0$. These Indices of x , when selected from the general Table, with their respective Parallelograms, will stand thus:

4						
3				$4m+3$		
		$2m+2$				
					$5m+1$	
						$6m$

Here if we would have an ascending Series for the Root y , we may apply the Ruler to the three external Terms 3 , $2m+2$, $6m$, which being made equal to each other, will give $m = \frac{1}{2}$, and each of the Numbers will be 3 . The Ruler in its parallel motion will next arrive at $5m+1$, or $3\frac{1}{2}$; then at 4 ; then at $4m+3$, or 5 ; which Numbers will be comprehended in the Arithmetical Progression $3, 3\frac{1}{2}, 4, 4\frac{1}{2}, 5$, whose common difference is $\frac{1}{2}$. This therefore will be the common difference of the Progression of the Indices, in the Series to be derived for y . So that now we intirely know the form of the Series, which will result from this Case. For if A, B, C, D , &c. be put to represent the several Coefficients of the Series in order, and as the first Index m is found to be $\frac{1}{2}$, and the common difference of the ascending Series is also $\frac{1}{2}$, we shall have here $y = Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2$, &c.

As to the Value of the first Coefficient A , this is found by putting the initial or external Terms of the Parallelogram equal to nothing.

D d

This

This here will give the Equation $A^6 - 7a^2A^2 + 6a^3 = 0$, which has these fix Roots, $A = \pm \sqrt{a}$, $A = \pm \sqrt{2a}$, $A = \pm \sqrt{-3a}$, of which the two last are impossible, and to be rejected. Of the others any one may be taken for A , according as we would prosecute this or that Root of the Equation.

Now that this is a legitimate Method for finding the first Approximation Ax^m , may appear from considering, that when the Terms of the Equation are thus ranged, according to a double Arithmetical Scale, the initial or external Terms, (each Case in its turn,) become the most considerable of the Series, and the rest continually decrease, or become of less and less value, according as they recede more and more from those initial Terms. Consequently they may be all rejected, as least considerable, which will make those initial or external Terms to be (nearly) equal to nothing; which Supposition gives the Value of A , or of Ax^m , for the first Approximation. And this Supposition is afterwards regularly pursued in the subsequent Operations, and proper Supplements are found, by means of which the remaining Terms of the Root are extracted.

We may try here likewise, if we can obtain a descending Series for the Root y , by applying the Ruler to the two external Terms $4m + 3$ and $6m$; which being made equal to each other, will give $m = \frac{3}{2}$, and hence each of the Numbers will be 9. The Ruler in its motion will next arrive at $5m + 1$, or $8\frac{1}{2}$. Then at $2m + 2$, or 5. Then at 4. And lastly at 3. But these Numbers 9, $8\frac{1}{2}$, 5, 4, 3, will be comprehended in an Arithmetical Progression, of which the common difference is $\frac{1}{2}$. So that the form of the Series here will be $y = Ax^{\frac{3}{2}} + Bx + Cx^{\frac{1}{2}} + Dx^0$, &c. But if we put the two external Terms equal to nothing, in order to obtain the first Approximation, we shall have $A^6 + \frac{A^4}{a} = 0$, or $A^2 + \frac{1}{a} = 0$, which will afford none but impossible Roots. So that we can have no initial Approximation from this supposition, and consequently no Series.

But lastly, to try the third and last case of external Parallelograms, we may apply the Ruler to 4 and $4m + 3$, which being made equal, will give $m = \frac{1}{4}$, and each of the Numbers will be 4. The next Number will be 3; the next $2m + 2$, or $2\frac{1}{2}$; the next $5m + 1$, or $2\frac{1}{4}$; the last will be $6m$, or $1\frac{1}{2}$. But the Numbers 4, 3, $2\frac{1}{2}$, $2\frac{1}{4}$, $1\frac{1}{2}$, will all be found in a decreasing Arithmetical Progression, whose common difference will be $\frac{1}{4}$. So that $Ax^{\frac{1}{4}} + Bx^0 + Cx^{-\frac{1}{4}} + Dx^{-\frac{1}{2}}$, &c. may represent the form of this Series, if the circumstances of the

who would lay the blame of their own Absurdities upon the Analytical Art. But these Absurdities are not to be imputed to the Art, but rather to the unskilfulness of the Artist, who thus absurdly applies the Principles of his Art.) Having therefore $m = 0$, we shall also have the Numbers $2m = 5m = 0$. The Ruler in its parallel motion will next arrive at 2; and then at 3. But the Numbers 0, 2, 3, will be comprehended in the Arithmetical Progression 0, 1, 2, 3, whose common difference is 1. Therefore $y = A + Bx + Cx^2$, &c. will be the form of this Series. Now from the exterior Terms $A^3 - 6A^2 = 0$, or $A^3 = 6$, or $A = 6^{\frac{1}{3}}$, we shall have the first Term of the Series.

There is another case of external Terms to be try'd, which possibly may afford a descending Series for y . For applying the Ruler to the Parallelograms 3 and $5m$, and making these equal, we shall have $m = \frac{3}{5}$, and each of these Numbers will be 3. Then the Ruler will come to 2; and lastly $2m$, or $\frac{6}{5}$. But the Numbers 3, 2, $1\frac{1}{5}$, will be comprehended in a descending Progression, whose common difference is $\frac{1}{5}$. Therefore the form of the Series will be $y = Ax^{\frac{1}{5}} + Bx^{\frac{2}{5}} + Cx^{\frac{3}{5}} + D$, &c. And the external Terms $A^5 x^3 - x^3 = 0$ will give $A = 1$ for the first Coefficient. Now as the two former cases will each give a converging Series for y in this Equation, when x is less than Unity; so this case will afford us a Series when x is greater than Unity; which will converge so much the faster, the greater x is supposed to be.

32. We have already seen the form of this Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, when the Terms are disposed according to a double Arithmetical Scale. And if we take the fictitious quantity Ax^m to represent the first Approximation to the Root y , we shall have by substitution $A^3 x^{3m} + aAx^{m+1} + a^2 Ax^m - x^3 - 2a^3$, &c. $= 0$. These Terms, or at least these Indices of x , being selected out of the general Table, will appear thus.

Now to obtain an ascending Series for the Root y , we may apply the Ruler to the three external Terms 0, m , $3m$, which being made equal, will give $m = 0$. Therefore these Numbers are each 0. In the next place the Ruler will come to $m + 1$, or 1; and lastly

3			
	$m+1$		
0	m		$3m$

to 3. But the Numbers 0, 1, 3, are contain'd in the Arithmetical Progression 0, 1, 2, 3, whose common difference is 1. Therefore the form of the Root is $y = A + Bx + Cx^2 + Dx^3$, &c. Now if the Equation $A^3 + a^2 A - 2a^3 = 0$, (which is derived from the initial

initial Terms,) is divided by the factor $A^2 + aA + 2a^2$, it will give the Quotient $A - a = 0$, or $A = a$ for the initial Term of the Root y .

If we would also derive a descending Series for this Equation, we may apply the Ruler to the external Parallelograms 3, $3m$, which being made equal to each other, will give $m = 1$; also these Numbers will each be 3. Then the Ruler will approach to $m + 1$, or 2; then to m , or 1; lastly to 0. But the Numbers 3, 2, 1, 0, are a decreasing Arithmetical Progression, of which the common difference is 1. So that the form of the Series will here be $y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. And the Equation form'd by the external Terms will be $A^3x^3 - x^3 = 0$, or $A = 1$.

33. The form of the Equation $x^2y^5 - 3c^4xy^2 - c^5x^2 + c^7 = 0$, as express'd by a combined Arithmetical Scale, we have already seen, which will easily shew us all the varieties of external Terms, with their other Circumstances. But for farther illustration, putting Ax^m for the first Term of the Root y , we shall have by substitution $A^5x^{5m+2} - 3c^4A^2x^{2m+1} - c^5x^2 + c^7 = 0$. These Indices of x being tabulated, will stand thus.

Now to have an ascending Series, we must apply the Ruler to the two external Terms 0 and $5m + 2$, which being made equal, will give

2					$5m+2$
		$2m+1$			
0					

$m = -\frac{2}{5}$, and the two Numbers arising will be each 0. The next Number that the Ruler arrives at is $2m + 1$, or $\frac{1}{5}$; and the last is 2. But the Numbers 0, $\frac{1}{5}$, 2, will be found in an ascending Arithmetical Progression, whose common difference is $\frac{1}{5}$. Therefore $y = Ax^{-\frac{2}{5}} + Bx^{\frac{1}{5}} + C + Dx^{\frac{2}{5}}$, &c. will be the form of the Root. To determine the first Coefficient A , we shall have from the exterior Terms $A^5 + c^7 = 0$, which will give $A = -\sqrt[5]{c^7} = -c^{\frac{7}{5}}$. Therefore the first Term or Approximation to the Root will be $y = -\sqrt[5]{\frac{c^7}{x^2}}$, &c.

We may try if we can obtain a descending Series, by applying the Ruler to the two external Parallelograms, whose Numbers are 2 and $5m + 2$, which being made equal, will give $m = 0$, and these Numbers will each be 2. The Ruler will next arrive at $2m + 1$, or 1; and lastly at 0. But the Numbers 2, 1, 0, form a descending Progression, whose common difference is 1. So that the form of the Series will here be $y = A + Bx^{-1} + Cx^{-2}$, &c. And putting the initial

initial Terms equal to nothing, as they stand in the Equation, we shall have $A^3x^2 - c^3x^2 = 0$, or $A = c$, for the first Approximation to the Root. And this Series will be accommodated to the case of Convergency, when x is greater than c ; as the other Series is accommodated to the other case, when x is less than c .

34. If the proposed Equation be $8z^6y^3 + az^6y^2 - 27a^9 = 0$, it may be thus resolved without any preparation. When reduced to our form, it will stand thus, $8z^6y^3 + az^6y^2 * - 27a^9 * \} = 0$; and by putting $y = Az^m$, &c. it will become $8A^3z^{3m+6} + aA^2z^{2m+6} * - 27a^9 * \} = 0$.

The first case of external Terms will give $8A^3z^{3m+6} - 27a^9 = 0$, whence $3m + 6 = 0$, or $m = -2$. These Indices or Numbers therefore will be each 0; and the other $2m + 6$ will be 2. But 0, 2, will be in an ascending Arithmetical Progression, of which the common difference is 2. So that the form of the Series will be $y = Az^{-2} + B + Cz^2 + Dz^4$, &c. And because $8A^3 = 27a^9$, or $2A = 3a^3$, it will be $A = \frac{3}{2}a^3$. Therefore the first Term or Approximation to the Root will be $\frac{3a^3}{2z^2}$.

But another case of external Terms will give $aA^2z^{2m+6} - 27a^9 = 0$, whence $2m + 6 = 0$, or $m = -3$. These Indices or Numbers therefore will be each 0; and the other $3m + 6$ will be -3 . But 0, -3 , will be found in a descending Arithmetical Progression, whose common difference is 3. So that the form of the Series will be $y = Az^{-3} + Bz^{-6} + Cz^{-9}$, &c. And because $aA^2 = 27a^9$, 'tis $A = \pm 3\sqrt{3} \times a^4$, for the first Coefficient.

Lastly, there is another case of external Terms, which may possibly afford us a descending Series, by making $8A^3z^{3m+6} + aA^2z^{2m+6} = 0$; whence $m = 0$. And the Numbers will be each equal to 6; the other Number, or Index of z , is 0. But 6, 0, will be in a descending Arithmetical Progression, of which the common difference is 6. Therefore the form of the Series will be $y = A + Bz^{-6} + Cz^{-12}$, &c. Also because $8A^3 + aA^2 = 0$, it is $A = -\frac{1}{8}a$ for the first Coefficient.

I shall produce one Example more, in order to shew what variety of Series may be derived from the Root in some Equations; as also to shew all the cases, and all the varieties that can be derived, in the present state of the Equation. Let us therefore assume this Equation, $y^3 - \frac{y^2x^2}{a} + x^3 - \frac{a^3x^2}{y^2} + \frac{a^6}{y^3} - \frac{a^7}{y^2x^2} + \frac{a^6}{x^3} - \frac{a^3y^2}{x^2} + a^3 = 0$, or rather $y^3 - a^{-1}y^2x^2 + x^3 - a^3y^{-2}x^2 + a^6y^{-3} - a^7y^{-2}x^{-2} + a^6x^{-3} - a^3y^2x^{-2} + a^3 = 0$. Which if we make $y = Ax^m$, &c. and dispose

dispose the Terms according to a combined Arithmetical Progression, will appear thus :

$$\begin{array}{ccccccc}
 & & & + a^6 x^{-3} & & & \\
 - a^3 A^2 x^{2m-2} & * & & * & * & - a^7 A^{-2} x^{-2m-2} & \\
 & * & * & * & * & * & \\
 A^3 x^{3m} & * & * & + a^3 & * & * & + a^6 A^{-3} x^{-3m}, \&c. \\
 & * & * & * & * & * & \\
 - a^{-1} A^2 x^{2m+2} & * & & * & * & - a^3 A^{-2} x^{-2m+2} & \\
 & & & + x^3 & & &
 \end{array} \Bigg\} = 0.$$

Now here it is plain by the disposition of the Terms, that the Ruler can be apply'd eight times, and no oftner, or that there are eight cases of external Terms to be try'd, each of which may give a Series for the Root, if the Coefficients will allow it, of which four will be ascending, and four descending. And first for the four cases of ascending Series, in which the Root will converge by the ascending Powers of x ; and afterwards for the other four cases, when the Series converges by the descending Powers of x .

I. Apply the Ruler, or, (which is the same thing,) assume the Equation $a^6 A^{-3} x^{-3m} - a^7 A^{-2} x^{-2m-2} = 0$, which will give $-3m = -2m - 2$, or $m = 2$; also $A = \frac{1}{a}$. The Number resulting from these Indices is -6 . But the Ruler in its parallel motion will next come to the Index -3 ; then to $-2m + 2$, or -2 ; then to 0 ; then to $2m - 2$, or 2 ; then to 3 ; and lastly to $3m$ and $2m + 2$, or 6 . But the Numbers $-6, -3, -2, 0, 2, 3, 6$, are in an ascending Arithmetical Progression, of which the common difference is 1 ; and therefore the form of the Series will be $y = Ax^2 + Bx^3 + Cx^4, \&c.$ and its first Term will be $\frac{x^2}{a}$.

II. Assume the Equation $a^6 x^{-3} - a^7 A^{-2} x^{-2m-2} = 0$, which will give $-3 = -2m - 2$, or $m = \frac{1}{2}$; also $A = \pm a^{\frac{1}{2}}$. The Number resulting hence is -3 ; the next will be $-3m$, or $-1\frac{1}{2}$; the next $2m - 2$, or -1 ; the next 0 ; the next $-2m + 2$, or 1 ; the next $3m$, or $1\frac{1}{2}$; the two last $2m + 2$ and 3 , are each 3 . But the Numbers $-3, -1\frac{1}{2}, -1, 0, 1, 1\frac{1}{2}, 3$, will be found in an ascending Arithmetical Progression, of which the common difference is $\frac{1}{2}$; and therefore the form of the Series will be $y = Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2, \&c.$ and its first Term will be $\pm \sqrt{ax}$.

III.

III. Assume the Equation $a^6 x^{-3} - a^3 A^2 x^{2m-2} = 0$, which will give $-3 = 2m - 2$, or $m = -\frac{1}{2}$; also $A = \pm a^{\frac{3}{2}}$. The Number resulting is -3 ; the next $3m$, or $-1\frac{1}{2}$; the next $-2m - 2$, or -1 ; the next 0 ; the next $2m + 2$, or 1 ; the next $-3m$, or $1\frac{1}{2}$; the two last 3 and $-2m + 2$, which are each 3 . But the Numbers $-3, -1\frac{1}{2}, -1, 0, 1, 1\frac{1}{2}, 3$, will be all comprehended in an ascending Arithmetical Progression, of which the common difference is $\frac{1}{2}$; and therefore the form of the Series will be $y = Ax^{-\frac{1}{2}} + B + Cx^{\frac{1}{2}} + Dx$, &c. and the first Term will be $\pm a^{\frac{3}{2}} x^{-\frac{1}{2}}$, or $\pm a \sqrt{\frac{a}{x}}$.

IV. Assume the Equation $A^3 x^{3m} - a^3 A^2 x^{2m-2} = 0$, which will give $3m = 2m - 2$, or $m = -2$; also $A = a^3$. The Number resulting is -6 ; the next will be -3 ; the next $2m + 2$, or -2 ; the next 0 ; the next $-2m - 2$, or 2 ; the next 3 ; the two last $-3m$ and $-2m + 2$, each of which is 6 . But the Numbers $-6, -3, -2, 0, 2, 3, 6$, belong to an ascending Arithmetical Progression, of which the common difference is 1 . Therefore the form of the Series will be $y = Ax^{-2} + Bx^{-1} + C + Dx$, &c. and its first Term will be $\frac{a^3}{x^2}$.

The four descending Series are thus derived.

I. Assume the Equation $A^3 x^{3m} - a^{-1} A^2 x^{2m+2} = 0$, which will give $3m = 2m + 2$, or $m = 2$; also $A = \frac{1}{a}$. The Number resulting is 6 ; the next will be 3 ; the next $2m - 2$, or 2 ; the next 0 ; the next $-2m + 2$, or -2 ; the next -3 ; the two last $-3m$ and $-2m - 2$, each of which is -6 . But the Numbers $6, 3, 2, 0, -2, -3, -6$, belong to a descending Arithmetical Progression, of which the common difference is 1 . Therefore the form of the Series will be $y = Ax^2 + Bx + C + Dx^{-1}$, &c. and the first Term will be $\frac{x^2}{a}$.

II. Assume the Equation $x^3 - a^{-1} A^2 x^{2m+2} = 0$, which will give $2m + 2 = 3$, or $m = \frac{1}{2}$; also $A = \pm a^{\frac{1}{2}}$. The Number resulting is 3 ; the next will be $3m$, or $1\frac{1}{2}$; the next $-2m + 2$, or 1 ; the next 0 ; the next $2m - 2$, or -1 ; the next $-3m$, or $-1\frac{1}{2}$; the two last -3 and $-2m - 2$ are each -3 . But the Numbers $3, 1\frac{1}{2}, 1, 0, -1, -1\frac{1}{2}, -3$, belong to a descending Arithmetical Progression, of which the common difference is $\frac{1}{2}$. Therefore the form of the Series will be $y = Ax^{\frac{1}{2}} + Bx^0 + Cx^{-\frac{1}{2}} + Dx^{-1}$, &c. and the first Term will be $\pm \sqrt{ax}$.

III.

III. Assume the Equation $x^3 - a^2 A^{-2} x^{-2m+2} = 0$, which will give $3 = -2m + 2$, or $m = -\frac{1}{2}$; also $A = \pm a^2$. The Number resulting from hence is 3; the next will be $-3m$, or $1\frac{1}{2}$; the next $2m + 2$, or 1; the next 0; the next $-2m - 2$, or -1 ; the next $3m$, or $-1\frac{1}{2}$; the two last -3 and $2m - 2$, each of which are -3 . But the Numbers 3, $1\frac{1}{2}$, 1, 0, -1 , $-1\frac{1}{2}$, -3 , are comprehended in a descending Arithmetical Progression, of which the common difference is $\frac{1}{2}$. Therefore the form of the Series will be $y = Ax^{-\frac{1}{2}} + Bx^{-1} + Cx^{-\frac{3}{2}} + Dx^{-2}$, &c. and the first Term will be $\pm a^2 x^{-\frac{1}{2}}$, or $\pm a \sqrt{x}$.

IV. Lastly, assume the Equation $a^6 A^{-3} x^{-3m} - a^3 A^{-2} x^{-2m+2} = 0$, which will give $-3m = -2m + 2$, or $m = -2$; also $A = a^3$. The Number resulting is 6; the next will be 3; the next $-2m - 2$, or 2; the next 0; the next $2m + 2$, or -2 ; the next -3 ; the two next $3m$ and $2m - 2$, are each -6 . But the Numbers 6, 3, 2, 0, -2 , -3 , -6 , belong to a descending Arithmetical Progression, of which the common difference is 1. Therefore the form of the Series will be $y = Ax^{-2} + Bx^{-3} + Cx^{-4} + Dx^{-5}$, &c. and the first Term is $\frac{a^3}{x^2}$.

And this may suffice in all Equations of this kind, for finding the forms of the several Series, and their first Approximations. Now we must proceed to their farther Resolution, or to the Method of finding all the rest of the Terms successively.

SECT. V. *The Resolution of Affected Specious Equations, prosecuted by various Methods of Analysis.*

35. **H**ITHERTO it has been shewn, when an Equation is proposed, in order to find its Root, how the Terms of the Equation are to be disposed in a two-fold regular succession, so as thereby to find the initial Approximations, and the several forms of the Series in all their various circumstances. Now the Author proceeds in like manner to discover the subsequent Terms of the Series, which may be done with much ease and certainty, when the form of the Series is known. For this end he finds Residual or Supplemental Equations, in a regular succession also, the Roots of which are a continued Series of Supplements to the Root required. In every one of which Supplemental Equations the Approximation is found,

found, by rejecting the more remote or less considerable Terms, and so reducing it to a simple Equation, which will give a near Value of the Root. And thus the whole affair is reduced to a kind of Comparison of the Roots of Equations, as has been hinted already. The Root of an Equation is nearly found, and its Supplement, which should make it compleat, is the Root of an inferior Equation, the Supplement of which is again the Root of an inferior Equation; and so on for ever. Or retaining that Supplement, we may stop where we please.

36. The Author's Diagram, or his Process of Resolution, is very easy to be understood; yet however it may be thus farther explain'd. Having inserted the Terms of the given Equation in the left-hand Column, (which therefore are equal to nothing, as are also all the subsequent Columns,) and having already found the first Approximation to the Root to be a ; instead of the Root y he substitutes its equivalent $a + p$ in the several Terms of the Equation, and writes the Result over-against them respectively, in the right-hand Margin. These he collects and abbreviates, writing the Result below, in the left-hand Column; of which rejecting all the Terms of too high a composition, he retains only the two lowest Terms $4a^2p + a^2x = 0$, which give $p = -\frac{1}{4}x$ for the second Term of the Root. Then assuming $p = -\frac{1}{4}x + q$, he substitutes this in the descending Terms to the left-hand, and writes the Result in the Column to the right-hand. These he collects and abbreviates, writing the Result below in the left-hand Column. Of which rejecting again all the higher Terms, he retains only the two lowest $4a^2q - \frac{1}{8}ax^2 = 0$, which give $q = \frac{x^2}{64a}$ for the third Term of the Root. And so on.

Or in imitation of a former Process, (which may be seen, pag. 165.) the Resolution of this, and all such like Equations, may be thus perform'd.

$$\begin{array}{l} y^3 + a^2y = 2a^3 = (\text{if } y = a + p) \left. \begin{array}{l} a^3 + 3a^2p + 3ap^2 + p^3 \\ + axy + x^3 \\ + a^2x + axp \end{array} \right\} \begin{array}{l} \text{Or collecting} \\ \text{and expung-} \\ \text{ing,} \end{array} \end{array}$$

$$\begin{array}{l} 4a^2p + 3ap^2 + p^3 = -a^2x = (\text{if } p = -\frac{1}{4}x + q) \left. \begin{array}{l} -a^2x + 4a^2q \\ -\frac{1}{4}ax^2 + axq \\ + \frac{3}{8}ax^2 - \frac{3}{2}axq + 3aq^2 \\ -\frac{1}{64}x^3 + \frac{3}{16}x^2q - \frac{3}{4}xq^2 + q^3 \end{array} \right\} \end{array}$$

Or collecting and expunging,

$$\begin{array}{l} 4a^2q + 3aq^2 + q^3 = \frac{1}{8}ax^2 = (\text{if } q = \frac{x^2}{64a} + r) \&c. \\ -\frac{1}{4}axq - \frac{3}{4}xq^2 + \frac{6}{8}x^2q \end{array}$$

By which Process the Root will be found $y = a - \frac{1}{4}x + \frac{x^2}{64a} \&c.$ Or

Or in imitation of the Method before taught, (pag. 178, &c.) we may thus resolve the first Supplemental Equation of this Example; viz. $4a^2p + axp + 3ap^2 + p^3 = -a^2x + x^3$; where the Terms must be dispos'd in the following manner. But to avoid a great deal of unnecessary prolixity, it may be here observed, that $y = a$, &c. briefly denotes, that a is the first Term of the Series, to be derived for the Value of y . Also $y = * - \frac{1}{4}x$, &c. insinuates, that $-\frac{1}{4}x$ is the second Term of the same Series y . Also $y = * * + \frac{x^2}{64a}$, &c. insinuates, that $+\frac{x^2}{64a}$ is the third Term of the Series y , without any regard to the other Terms. And so for all the succeeding Terms; and the like is to be understood of all other Series whatever.

$$\left. \begin{array}{l} 4a^2p \\ + axp \\ + 3ap^2 \\ + p^3 \end{array} \right\} = \begin{array}{l} -a^2x \\ + \frac{1}{8}ax^2 - \frac{3x^3}{128} + \frac{509x^4}{4096a}, \text{ \&c.} \\ -\frac{1}{4}ax^2 + \frac{1}{8}x^3 + \frac{131x^4}{512a}, \text{ \&c.} \\ +\frac{3}{8}ax^2 - \frac{3x^3}{128} - \frac{1569x^4}{4096a}, \text{ \&c.} \\ -\frac{1}{8}x^3 + \frac{131x^4}{1024a}, \text{ \&c.} \end{array}$$

$$p = -\frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}, \text{ \&c.}$$

To explain this Process, it may be observed, that here $-a^2x$ is made the first Term of the Series, into which $4a^2p$ is to be resolved; or $4a^2p = -a^2x$, &c. and therefore $p = -\frac{1}{4}x$, &c. which is set down below. Then is $+axp = -\frac{1}{4}ax^2$, &c. and (by squaring) $+3ap^2 = +\frac{3}{8}ax^2$, &c. each of which are set down in their proper Places. These Terms being collected, will make $-\frac{1}{8}ax^2$, which with a contrary Sign must be set down for the second Term of $4a^2p$; or $4a^2p = * + \frac{1}{8}ax^2$, &c. and therefore $p = * + \frac{x^2}{64a}$, &c. Then $axp = * + \frac{x^3}{64a}$, &c. and (by squaring) $3ap^2 = * - \frac{3x^3}{128}$, &c. and (by cubing) $p^3 = -\frac{1}{8}x^3$, &c. These being collected will make $-\frac{3x^3}{128}$, to be wrote down with a contrary Sign; and this, together with x^3 , one of the Terms of the given Equation, will make $4a^2p = * * + \frac{131x^3}{128a^2}$, &c. and therefore $p = * * + \frac{131x^3}{512a^2}$, &c. Then $axp = * * + \frac{131x^4}{512a^3}$, &c. and (by squaring) $3ap^2 = * *$

$\frac{1569x^4}{4096a}$, &c. and (by cubing) $p^3 = * + \frac{3x^4}{1024a}$, &c. all which being collected with a contrary Sign, will make $4a^2p = *** + \frac{509x^4}{16384a}$, &c. and therefore $p = *** + \frac{509x^4}{16384a}$, &c. And by the same Method we may continue the Extraction as far as we please.

The *Rationale* of this Process has been already deliver'd, but as it will be of frequent use, I shall here mention it again, in somewhat a different manner. The Terms of the Equation being duly order'd, so as that the Terms involving the Root, (which are to be resolv'd into their respective Series,) being all in a Column on one side, and the known Terms on the other side; any adventitious Terms may be introduced, such as will be necessary for forming the several Series, provided they are made mutually to destroy one another, that the integrity of the Equation may be thereby preserved. These adventitious Terms will be supply'd by a kind of Circulation, which will make the work easy and pleasant enough; and the necessary Terms of the simple Powers or Roots, of such Series as compose the Equation, must be derived one by one, by any of the foregoing Theorems.

Or if we are willing to avoid too many, and too high Powers in these Extractions, we may proceed in the following manner. The Example shall be the same Supplemental Equation as before, which may be reduced to this form, $4a^2 + ax + 3ap + pp \times p = -a^2x + x^3$, of which the Resolution may be thus:

$$\begin{array}{r}
 4a^2 + ax \\
 + 3ap = -\frac{3}{4}ax + \frac{3}{8}x^2 + \frac{393x^3}{512a}, \text{ \&c.} \\
 + p^3 = -\frac{1}{8}x^2 - \frac{x^3}{128a}, \text{ \&c.} \\
 \hline
 4a^2 + \frac{1}{4}ax + \frac{7}{8}x^2 + \frac{389x^3}{512a}, \text{ \&c.} \\
 p = -\frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}, \text{ \&c.} \\
 \hline
 -a^2x + x^3
 \end{array}$$

The Terms $4a^2 + ax + 3ap + pp$ I call the aggregate Factor, of which I place the known part or parts $4a^2 + ax$ above, and the unknown parts $3ap + pp$ in a Column to the left-hand, so as that their respective Series, as they come to be known, may be placed regularly over-against them. Under these a Line is drawn, to receive the

the aggregate Series beneath it, which is form'd by the Terms of the aggregate Factor, as they become known. Under this aggregate Series comes the simple Factor p , or the symbol of the Root to be extracted, as its Terms become known also. Lastly, under all are the known Terms of the Equation in their proper places. Now as these last Terms (because of the Equation) are equivalent to the Product of the two Species above them; from this consideration the Terms of the Series p are gradually derived, as follows.

First, the initial Term $4a^2$ (of the aggregate Series) is brought down into its place, as having no other Term to be collected with it. Then because this Term, multiply'd by the first Term of p , suppose q , is equal to the first Term of the Product, that is, $4a^2q = -a^2x$, it will be $q = -\frac{1}{4}x$, or $p = -\frac{1}{4}x$, &c. to be put down in its place. Thence we shall have $3ap = -\frac{3}{4}ax$, &c. which together with $+\frac{1}{4}ax$ above, will make $+\frac{1}{4}ax$ for the second Term of the aggregate Series. Now if we suppose r to represent the second Term of p , and to be wrote in its place accordingly; by cross-multiplication we shall have $4a^2r - \frac{1}{6}ax^2 = 0$, because the second Term of the Product is absent, or $= 0$. Therefore $r = \frac{x^2}{64a}$, which may now be set down in its place. And hence $3ap = * + \frac{3}{84}x^2$, &c. and $p^2 = -\frac{1}{6}x^2$, &c. which being collected will make $-\frac{7}{64}x^2$, for the third Term of the aggregate Factor. Now if we suppose s to represent the third Term of p , then by cross-multiplication, (or by our Theorem for Multiplication of infinite Series,) $4a^2s + \frac{x^3}{256} - \frac{7x^3}{256} = x^3$; (for x^3 is the third Term of the Product.) Therefore $s = \frac{131x^3}{512a^2}$, to be set down in its place. Then $3ap = ** + \frac{393x^3}{512a}$, &c. and $p^2 = * - \frac{x^3}{128a}$, &c. which together will make $+\frac{389x^3}{512a}$ for the fourth Term of the aggregate Series. Then putting t to represent the fourth Term of p , by multiplication we shall have $4a^2t + \frac{131x^4}{2048a} + \frac{7x^4}{4096a} - \frac{389x^4}{2048a} = 0$, whence $t = \frac{599x^4}{16384a^3}$, to be set down in its place. If we would proceed any farther in the Extraction, we must find in like manner the fourth Term of the Series $3ap$, and the third Term of p^2 , in order to find the fifth Term of the aggregate Series. And thus we may easily and surely carry on the Root to what degree of accuracy we please, without any danger of computing any superfluous Terms; which will be no mean advantage of these Methods.

Or

mon and obvious enough, but which supposes the form of the Series required to be already known, and only the Coefficients to be unknown. This we may the better do here, because we have already shewn how to determine the form and number of such Series, in any case proposed. This Method consists in the assumption of a general Series for the Root, such as may conveniently represent it, by the substitution of which in the given Equation, the general Coefficients may be determined. Thus in the present Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, having already found (pag. 204.) the form of the Root or Series to be $y = A + Bx + Cx^2, \&c.$ by the help of any of the Methods for Cubing an infinite Series, we may easily substitute this Series instead of y in this Equation, which will then become

$$\left. \begin{aligned} &A^3 + 3A^2Bx + 3AB^2x^2 + B^3x^3 + 3AC^2x^4, \&c. \\ &\quad + 3A^2C + 6ABC + 3B^2C \\ &\quad + 3A^2D + 6ABD \\ &\quad + 3A^2E \\ &+ aAx + aBx^2 + aCx^3 + aDx^4, \&c. \\ &+ a^2A + a^2Bx + a^2Cx^2 + a^2Dx^3 + a^2Ex^4, \&c. \\ &- 2a^3 \end{aligned} \right\} = 0.$$

Now because x is an indeterminate quantity, and must continue so to be, every Term of this Equation may be separately put equal to nothing, by which the general Coefficients $A, B, C, D, \&c.$ will be determined to congruous Values; and by this means the Root y will be known. Thus, (1.) $A^3 + a^2A - 2a^3 = 0$, which will give $A = a$, as before. (2.) $3A^2B + aA + a^2B = 0$, or $B = \frac{-aA}{3A^2 + a^2} = -\frac{1}{4}a$.

(3.) $3AB^2 + 3A^2C + aB + a^2C = 0$, or $C = -\frac{3AB^2 + aB}{3A^2 + a^2} = \frac{1}{64a}$.

(4.) $B^3 + 6ABC + 3A^2D + aC + a^2D - 1 = 0$, or $D = \frac{131}{512a^2}$.

(5.) $3AC^2 + 3B^2C + 6ABD + 3A^2E + aD + a^2E = 0$, or $E = \frac{509}{16384a^3}$. And so on, to determine $F, G, H, \&c.$ Then by substituting these Values of $A, B, C, D, \&c.$ in the assumed Root, we

shall have the former Series $y = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}, \&c.$

Or lastly, we may conveniently enough resolve this Equation, or any other of the same kind, by applying it to the general Theorem, pag. 190. for extracting the Roots of any affected Equations in Numbers. For this Equation being reduced to this form $y^3 + a^2 + ax$

$\times y - 2a^3 + x^3 \times y^0 = 0$, we shall have there $m = 3$. And instead of the first, second, third, fourth, fifth, &c. Coefficients of the Powers of y in the Theorem, if we write $1, 0, aa + ax, -2a^3 - x^3, 0$, &c. respectively; and if we make the first Approximation $\frac{A}{B} = \frac{a}{1}$, or $A = a$ and $B = 1$; we shall have $\frac{4a^3 + x^3}{4a^2 + ax}$ for a nearer Approximation to the Root. Again, if we make $A = 4a^3 + x^3$, and $B = 4a^2 + ax$, by Substitution we shall have the Fraction $\frac{256a^9 + 96a^8x + 24a^7x^2 + 114a^6x^3 + 48a^5x^4 + 12a^4x^5 + 25a^3x^6 + \dots + 2x^9}{256a^8 + 160a^7x + 60a^6x^2 + 109a^5x^3 + 25a^4x^4 + 12a^3x^5 + 3ax^7}$ for a nearer Approximation to the Root. And taking this Numerator for A , and the Denominator for B , we shall approach nearer still. But this last Approximation is so near, that if we only take the first five Terms of the Numerator, and divide them by the first five Terms of the Denominator, (which, if rightly managed, will be no troublesome Operation,) we shall have the same five Terms of the Series, so often found already.

And the Theorem will converge so fast on this, and such like occasions, that if we here take the first Approximation $A = a$, (making $B = 1$,) we shall have $y = \frac{4a^3 + x^3}{4a^2 + ax}$, &c. $= a - \frac{1}{4}x$, &c. And if again we make this the second Approximation, or $A = a - \frac{1}{4}x$, (making $B = 1$,) we shall have $y = \frac{2A^3 + 2a^3 + x^3}{3A^2 + a^2 + ax}$, &c. $= \frac{4a^3 - \frac{3}{2}a^2x + \frac{3}{8}ax^2 + \frac{13}{32}x^3}{4a^2 - \frac{1}{2}ax + \frac{1}{16}x^2}$, &c. $= a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. And if again we make this the third Approximation, or $A = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}$, &c. (making $B = 1$,) we shall have the Value of the true Root to eight Terms at this Operation. For every new Operation will double the number of Terms, that were found true by the last Operation.

To proceed still with the same Equation; we have found before, pag. 205, that we might likewise have a descending Series in this form, $y = Ax + B + Cx^{-1}$, &c. for the Root y , which we shall extract two or three ways, for the more abundant exemplification of this Doctrine. It has been already found, that $A = 1$, or that x is the first Approximation to the Root. Make therefore $y = x + p$, and substitute this in the given Equation $y^3 + axy + aay - x^3 - 2a^3 = 0$, which will then become $3x^2p + axp + a^2p + 3xp^2 + p^3 + ax^2 + a^2x - 2a^3 = 0$. This may be reduced to this form $3x^2 + ax + a^2 + 3xp + p^2 \times p = -ax^2 - a^2x + 2a^3$, and may be resolved as follows.

$$\begin{array}{r}
 3x^2 + ax + a^2 \quad * \\
 + 3xp - \dots - ax - a^2 + \frac{55a^3}{27x}, \text{ \&c.} \\
 + p^2 - \dots - + \frac{1}{9}a^2 + \frac{2a^3}{9x}, \text{ \&c.} \\
 \hline
 3x^2 \quad * + \frac{1}{9}a^2 + \frac{61a^3}{27x}, \text{ \&c.} \\
 p = -\frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2} + \frac{64a^4}{243x^3}, \text{ \&c.} \\
 \hline
 -ax^2 - a^2x + 2a^3 \quad *
 \end{array}$$

The Terms of the aggregate Factor, as also the known Terms of the Equation, being disposed as in the Paradigm, bring down $3x^2$ for the first Term of the aggregate Series; and supposing q to represent the first Term of the Series p , it will be $3x^2q = -ax^2$, or $q = -\frac{1}{3}a$, for the first Term of p . Therefore $-ax$ will be the first Term of $3xp$, to be put down in its place. This will make the second Term of the aggregate Series to be nothing; so that if r represent the second Term of p , we shall have by multiplication $3x^2r = -a^2x$, or $r = -\frac{a^2}{3x}$ for the second Term of p , to be put down in its place. Then will $-a^2$ be the second Term of $3xp$, as also $\frac{1}{9}a^2$ will be the first Term of p^2 , to be set down each in their places. The Result of this Column will be $\frac{2}{9}a^2$, which is to be made the third Term of the aggregate Series. Then putting s for the third Term of p , we shall have by Multiplication $3x^2s - \frac{1}{3}a^3 = 2a^3$, or $s = \frac{55a^3}{81x^2}$. And thus by the next Operation we shall have $t = \frac{64a^4}{243x^2}$, and so on.

Or if we would resolve this residual Equation by one of the foregoing Methods, by which the raising of Powers was avoided, and wherein the whole was perform'd by Multiplication alone; we may reduce it to this form, $3x^2 + ax + a^2 + 3x + p \times p \times p = -ax^2 - a^2x + 2a^3$, the Resolution of which will be thus:

F f

 $3x^2$

$$\begin{array}{r}
 3x^2 + ax + a^2 \quad * \\
 + \overline{3x + p} \quad - - - - + 3x - \frac{1}{3}a - \frac{a^2}{3x}, \&c. \\
 \times p \quad - - - - - \frac{1}{3}a + \frac{a^2}{3x} + \frac{55a^3}{81x^2}, \&c. \\
 \hline
 3x^2 \quad * + \frac{1}{9}a^2 + \frac{61a^3}{27x}, \&c. \\
 \times p = -\frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2} + \frac{64a^4}{243x^3}, \&c. \\
 \hline
 -ax^2 - a^2x + 2a^3 \quad *
 \end{array}$$

The Terms being dispos'd as in the Example, bring down $3x^2$ for the first Term of the aggregate Series, and supposing q to represent the first Term of the Series p , it will be $3x^2q = -ax^2$, or $q = -\frac{1}{3}a$. Put down $+3x$ in its proper place, and under it (as also after it) put down the first Term of p , or $-\frac{1}{3}a$, which being multiply'd, and collected with $+ax$ above, will make 0 for the second Term of the aggregate Series. If the second Term of p is now represented by r , we shall have $3x^2r = -a^2x$, or $r = -\frac{a^2}{3x}$, to be put down in its several places. Then by multiplying and collecting we shall have $+\frac{1}{9}a^2$ for the third Term of the aggregate Series. And putting s for the third Term of p , we shall have by Multiplication $3x^2s - \frac{1}{27}a^3 = 2a^3$, or $s = \frac{55a^3}{81x^2}$. And so on as far as we please.

Lastly, instead of the Supplemental Equation, we may resolve the given Equation itself in the following manner:

$$\begin{array}{r}
 y^3 \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} = x^3 \quad * \quad * \quad + 2a^3 \quad * \\
 \quad \quad \quad - ax^2 - \frac{2}{3}a^2x + \frac{2}{3}a^3 - \frac{28a^4}{81x}, \&c. \\
 +axy \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} - - - + ax^2 - \frac{1}{3}a^2x - \frac{1}{3}a^3 + \frac{55a^4}{81x}, \&c. \\
 +a^2y \quad \left. \begin{array}{l} \\ \end{array} \right\} - - - - - + a^2x - \frac{1}{3}a^3 - \frac{a^4}{3x}, \&c. \\
 \quad \quad \quad y = x - \frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2} + \frac{64a^4}{243x^3}, \&c.
 \end{array}$$

Here because it is $y^3 = x^3$, &c. it will be $y = x$, &c. and therefore $+axy = +ax^2$, &c. which must be set down in its place. Then it must be wrote again with a contrary sign, that it may be $y^3 = -ax^2$, &c. and therefore (extracting the cube-root,) $y = -\frac{1}{3}a$, &c. Then $+a^2y = +a^2x$, &c. and $+axy = -\frac{1}{3}a^2x$, &c. which

which being collected with a contrary sign, will make $y^3 = * * - \frac{2}{3}a^2x$, &c. and (by Extraction) $y = * * - \frac{a^2}{3x}$, &c. Hence $+a^2y = * - \frac{1}{3}a^3$, &c. and $+axy = * * - \frac{1}{3}a^3$, &c. which being collected with a contrary sign, and united with $+2a^3$ above, will make $y^3 = * * * \frac{2}{3}a^3$, &c. whence (by Extraction) $y = * * * \frac{5ca^3}{81x^2}$, &c. Then $+a^2y = * * - \frac{a^4}{3x}$, &c. and $+axy = * * * + \frac{5ca^4}{81x}$, &c. which being collected with a contrary sign, will make $y^3 = * * * * - \frac{28a^4}{81x}$, &c. and then (by Extraction) $y = * * * * + \frac{64a^4}{243x^5}$, &c. And so on.

37, 38. I think I need not trouble the Learner, or myself, with giving any particular Explication (or Application) of the Author's Rules, for continuing the Quote only to such a certain period as shall be before determined, and for preventing the computation of superfluous Terms; because most of the Methods of Analysis here deliver'd require no Rules at all, nor is there the least danger of making any unnecessary Computations.

39. When we are to find the Root y of such an Equation as this, $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$, &c. $= z$, this is usually call'd *the Reversion of a Series*. For as here the Aggregate z is express'd by the Powers of y ; so when the Series is reverted, the Aggregate y will be express'd by the Powers of z . This Equation, as now it stands, supposes z (or the Aggregate of the Series) to be unknown, and that we are to approximate to it indefinitely, by means of the known Number y and its Powers. Or otherwise; the unknown Number z is equivalent to an infinite Series of decreasing Terms, express'd by an Arithmetical Scale, of which the known Number y is the Root. This Root therefore must be supposed to be less than Unity, that the Series may duly converge. And thence it will follow, that z also will be much less than Unity. This is usually call'd a Logarithmick Series, because in certain circumstances it expresses the Relation between the Logarithms and their Numbers, as will appear hereafter. If we look upon z as known, and therefore y as unknown, the Series must be reverted; or the Value of y must be express'd by a Series of Terms compos'd of the known Number z and its Powers. The Author's Method for reverting this Series will be very obvious from the consideration of his Diagram; and we shall meet with another Method hereafter, in another part of his Works. It will be sufficient therefore in this place, to perform it after the manner of some of the foregoing Extractions.

$$\begin{array}{lcl}
 y & \} & = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5, \&c. \\
 - \frac{1}{2}y^2 & \} & - - - - - \frac{1}{2}z^2 - \frac{1}{2}z^3 - \frac{7}{24}z^4 - \frac{1}{8}z^5, \&c. \\
 + \frac{1}{3}y^3 & \} & - - - - - + \frac{1}{3}z^3 + \frac{1}{2}z^4 + \frac{5}{12}z^5, \&c. \\
 - \frac{1}{4}y^4 & \} & - - - - - \frac{1}{4}z^4 - \frac{1}{2}z^5, \&c. \\
 + \frac{1}{5}y^5 & \} & - - - - - + \frac{1}{5}z^5, \&c. \\
 \&c. & &
 \end{array}$$

In this Paradigm the unknown parts of the Equation are set down in a descending order to the left-hand, and the known Number z is set down over-against y to the right-hand. Then is $y = z, \&c.$ and therefore $-\frac{1}{2}y^2 = -\frac{1}{2}z^2, \&c.$ which is to be set down in its place, and also with a contrary sign, so that $y = * + \frac{1}{2}z^2, \&c.$ And therefore (squaring) $-\frac{1}{2}y^2 = * - \frac{1}{2}z^3, \&c.$ and (cubing) $+\frac{1}{3}y^3 = +\frac{1}{3}z^3, \&c.$ which Terms collected with a contrary sign, make $y = * * + \frac{1}{6}z^3, \&c.$ And therefore (squaring) $-\frac{1}{2}y^2 = * * - \frac{7}{24}z^4, \&c.$ and (cubing) $+\frac{1}{3}y^3 = * + \frac{1}{2}z^4, \&c.$ and $-\frac{1}{4}y^4 = -\frac{1}{4}z^4, \&c.$ which Terms collected with a contrary sign, make $y = * * * + \frac{1}{24}z^4, \&c.$ Therefore $-\frac{1}{2}y^2 = * * * - \frac{1}{8}z^5, \&c.$ and $+\frac{1}{3}y^3 = * * + \frac{5}{12}z^5, \&c.$ and $-\frac{1}{4}y^4 = * - \frac{1}{2}z^5, \&c.$ and $+\frac{1}{5}y^5 = +\frac{1}{5}z^5, \&c.$ which Terms collected with a contrary sign, make $y = * * * * + \frac{1}{120}z^5, \&c.$ And so of the rest.

40. Thus if we were to revert the Series $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{1152}y^9 + \frac{63}{16384}y^{11}, \&c. = z,$ (where the Aggregate of the Series, or the unknown Number z , will represent the Arch of a Circle, whose Radius is 1, if its right Sine is represented by the known Number y ,) or if we were to find the value of y , consider'd as unknown, to be express'd by the Powers of z , now consider'd as known; we may proceed thus:

$$\begin{array}{lcl}
 y & \} & = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9, \&c. \\
 + \frac{1}{6}y^3 & \} & - - - + \frac{1}{6}z^3 - \frac{1}{12}z^5 + \frac{1}{720}z^7 - \frac{1}{18144}z^9, \&c. \\
 + \frac{3}{40}y^5 & \} & - - - - - + \frac{3}{40}z^5 - \frac{1}{160}z^7 + \frac{1}{9600}z^9, \&c. \\
 + \frac{5}{112}y^7 & \} & - - - - - + \frac{5}{112}z^7 - \frac{5}{672}z^9, \&c. \\
 + \frac{35}{1152}y^9 & \} & - - - - - + \frac{35}{1152}z^9, \&c. \\
 \&c. & &
 \end{array}$$

The Terms being disposed as you see here, we shall have $y = z, \&c.$ and therefore (cubing) $\frac{1}{6}y^3 = \frac{1}{6}z^3, \&c.$ which makes $y = * - \frac{1}{6}z^3, \&c.$ so that (cubing) we shall have $+\frac{1}{6}y^3 = * - \frac{1}{12}z^5, \&c.$ and also $\frac{3}{40}y^5 = \frac{3}{40}z^5, \&c.$ and collecting with a contrary sign,

$y = * * + \frac{1}{12} z^5, \&c.$ Hence $\frac{1}{6} y^3 = * * \frac{1}{72} z^7, \&c.$ and $\frac{3}{40} y^4 = * - \frac{1}{16} z^7, \&c.$ and $\frac{5}{112} y^5 = \frac{5}{112} z^7, \&c.$ and collecting with a contrary sign, $y = * * * - \frac{1}{304} z^7, \&c.$ And so on.

If we should desire to perform this Extraction by another of the foregoing Methods, that is, by supposing the Equation to be reduced to this form $1 + \frac{1}{6} y^2 + \frac{3}{40} y^4 + \frac{5}{112} y^6 + \frac{35}{1152} y^8, \&c. \times y = z$, it may be sufficient to set down the Praxis, as here follows.

$$\begin{array}{r}
 + \frac{1}{6} y^2 - - - + \frac{1}{6} z^2 - \frac{1}{18} z^4 + \frac{1}{72} z^6 - \frac{1}{1890} z^8, \&c. \\
 + \frac{3}{40} y^4 - - - + \frac{3}{40} z^4 - \frac{1}{20} z^6 + \frac{3}{800} z^8, \&c. \\
 + \frac{5}{112} y^6 - - - + \frac{5}{112} z^6 - \frac{5}{112} z^8, \&c. \\
 + \frac{35}{1152} y^8 - - - + \frac{35}{1152} z^8, \&c. \\
 \hline
 1 + \frac{1}{6} z^2 + \frac{7}{360} z^4 + \frac{31}{15120} z^6 + \frac{827}{604800} z^8, \&c. \\
 y = z - \frac{1}{6} z^3 + \frac{1}{120} z^5 - \frac{1}{5040} z^7 + \frac{1}{362880} z^9, \&c. \\
 \hline
 z \quad * \quad * \quad * \quad *
 \end{array}$$

41. The affected Cubick Equation, which the Author here assumes to be solved, has infinite Series for the Coefficients of the Powers of y ; and therefore its Terms being disposed (as is taught before) according to a double Arithmetical Scale, the Roots of each of which are y and z , it will stand as is represented here below. Or taking Az^m for the first Approximation to the Root y , and substituting it in the first Table, it will appear as is here set down in the second Table.

$$\left. \begin{array}{l}
 * \quad * \quad * \quad - 8 \\
 z^2 y^3 - z^2 y^2 + z^2 y + z^2 \\
 - \frac{1}{2} z^4 + z^4 - 2 z^4 - 4 z^4 \\
 + \frac{1}{2} z^6 - z^6 + 3 z^6 + 9 z^6 \\
 - \frac{1}{2} z^8 + z^8 - 4 z^8 - 16 z^8 \\
 \&c. \quad \&c. \quad \&c. \quad \&c.
 \end{array} \right\} = 0. \quad \left. \begin{array}{l}
 * \quad * \quad * \quad - 8 \\
 A^3 z^{3m+2} - A^2 z^{2m+2} + A z^{m+2} + z^2 \\
 - \frac{1}{2} A^3 z^{3m+4} + A^2 z^{2m+4} - 2 A z^{m+4} - 4 z^4 \\
 + \frac{1}{2} A^3 z^{3m+6} - A^2 z^{2m+6} + 3 A z^{m+6} + 9 z^6 \\
 \&c. \quad \&c. \quad \&c. \quad \&c.
 \end{array} \right\} = 0.$$

Now the only case of external Terms, to be discover'd by applying the Ruler, will give the Equation $A^3 z^{3m+2} - 8 = 0$, whence $3m + 2 = 0$, or $m = -\frac{2}{3}$, and the Coefficient $A = 2$. The next Number or Index, to which the Ruler in its parallel motion will apply itself, will be $2m + 2$, or $\frac{2}{3}$; the next will be $m + 2$, or $\frac{4}{3}$; and so on. Which ascending Arithmetical Progression $0, \frac{2}{3}, \frac{4}{3}, \&c.$ will have $\frac{2}{3}$ for its common difference. Therefore $y = Az^{-\frac{2}{3}} + B + Cz^{\frac{2}{3}} + Dz^{\frac{4}{3}} + Ez^2, \&c.$ will be the form of the Root in this Equation. It may be resolved by any of the foregoing Methods, but

but perhaps most readily by substituting the Value of y now found in the given Equation, and thence determining the general Coefficients as before. By which the Root will be found to be $y = 2z^{-\frac{2}{3}} + \frac{1}{3} - \frac{1}{9}z^{\frac{2}{3}} + \frac{57}{162}z^{\frac{4}{3}} - \frac{14}{81}z^2 + \frac{299}{1452}z^{\frac{8}{3}} + \frac{671}{26244}z^{\frac{10}{3}}, \&c.$

42. To resolve this affected Quadratick Equation, in which one of the Coefficients is an infinite Series; if we suppose $y = Ax^m$, &c. we shall have (by Substitution) the Equation as it stands here below.

Then by applying the Ruler, we shall have $-aAx^m + \frac{x^4}{4a^2} = 0$,

whence $m = 4$, and $A = \frac{1}{4a^3}$. The next Index, that the Ruler in its parallel motion will arrive at, is $m + 1$, or 5; the next is $m + 2$, or 6; &c. so that the common difference of the Progression is 1, and the Root may be represented by $y = Ax^4 + Bx^5 + Cx^6$, &c. which may be extracted as here follows.

$$\begin{array}{l}
 A^2 x^{2m} - aAx^m \quad \&c. \\
 * - Ax^{m+1} \quad * \\
 * - \frac{A}{a} x^{m+2} \quad * \\
 * - \frac{A}{a^2} x^{m+3} \quad * \\
 * - \frac{A}{a^3} x^{m+4} + \frac{x^4}{4a^2} \quad \&c.
 \end{array}
 \left. \vphantom{\begin{array}{l} A^2 x^{2m} - aAx^m \\ * - Ax^{m+1} \\ * - \frac{A}{a} x^{m+2} \\ * - \frac{A}{a^2} x^{m+3} \\ * - \frac{A}{a^3} x^{m+4} \end{array}} \right\} = 0.$$

$$\begin{array}{l}
 -ay \\
 -xy \\
 -\frac{x^2}{a}y \\
 -\frac{x^3}{a^2}y \\
 -\frac{x^4}{a^3}y \\
 +y^2
 \end{array}
 \left. \vphantom{\begin{array}{l} -ay \\ -xy \\ -\frac{x^2}{a}y \\ -\frac{x^3}{a^2}y \\ -\frac{x^4}{a^3}y \end{array}} \right\} = -\frac{x^4}{4a^2} + \frac{x^5}{4a^3} * * - \frac{x^8}{16a^6}, \&c.$$

$$\begin{array}{l}
 -\frac{x^5}{4a^3} + \frac{x^6}{4a^4} * * \\
 -\frac{x^6}{4a^4} + \frac{x^7}{4a^5} * \&c. \\
 -\frac{x^7}{4a^5} + \frac{x^8}{4a^6}, \&c. \\
 -\frac{x^8}{4a^6}, \&c. \\
 + \frac{x^8}{16a^6}, \&c.
 \end{array}$$

$$y = \frac{x^4}{4a^3} - \frac{x^5}{4a^4} * * + \frac{x^8}{16a^7}, \&c.$$

Here because it is $-ay = -\frac{x^4}{4a^2}$, &c. it will be $y = \frac{x^4}{4a^3}$, &c. Therefore $-xy = -\frac{x^5}{4a^3}$, &c. which wrote with a contrary Sign will make $-ay = * + \frac{x^5}{4a^3}$, and therefore $y = * - \frac{x^5}{4a^4}$, &c. Then $-xy = * + \frac{x^6}{4a^4}$, &c. and $-\frac{x^2}{a}y = -\frac{x^6}{4a^4}$, &c. which collected will destroy each other, and therefore $-ay = * * + 0$, &c. and consequently $y = * * + 0$, &c. &c.

But there is another case of external Terms, which will be discover'd by the Ruler, and which will give $A^2 x^{2m} - aAx^m = 0$, whence $m = 0$, and $A = a$. Here the Progression of the Indices will be 0, 1, 2, &c. so that $y = A + Bx + Cx^2$, &c. will be the form of the Series. And if this Root be prosecuted by any of the

Methods

Methods taught before, it will be found $y = a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{3x^4}{4a^3}, \&c.$

Now in the given Equation, because the infinite Series $a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3}, \&c.$ is a Geometrical Progression, and therefore is equal to $\frac{a^2}{a-x}$, as may be proved by Division; if we substitute this, the Equation will become $y^2 - \frac{a^2}{a-x}y + \frac{x^4}{4a^2} = 0$. And if we extract the square-root in the ordinary way, it will give $y = \frac{a^2 \pm \sqrt{a^6 - a^4x^4 + 2ax^5 - x^6}}{2a^2 - 2ax}$ for the exact Root. And if this Radical

be resolved, and then divided by this Denominator, the same two Series will arise as before, for the two Roots of this Equation. And this sufficiently verifies the whole Process.

43. In Series that are very remarkable, and of general use, the Law of Continuation (if not obvious) should be always assign'd, when that can be conveniently done; which renders a Series still more useful and elegant. This may commonly be discover'd in the Computation, by attending to the formation of the Coefficients, especially if we put Letters to represent them, and thereby keep them as general as may be, descending to particulars by degrees. In the Logarithmic Series, for instance, $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4, \&c.$ the Law of Consecution is very obvious, so that any Term, tho' ever so remote, may easily be assign'd at pleasure. For if we put T to represent any Term indefinitely, whose order in the Series is express'd by the natural Number m , then will $T = \pm \frac{1}{m}y^m$, where the Sign must be $+$ or $-$ according as m is an odd or an even Number. So that the hundredth Term is $-\frac{1}{100}y^{100}$, the next is $+\frac{1}{101}y^{101}, \&c.$ In the Reverse of this Series, or $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \frac{1}{120}z^5, \&c.$ the Law of Continuation is thus. Let T represent any Term indefinitely, whose order in the Series is express'd by m ; then is $T = \frac{z^m}{1 \times 2 \times 3 \times 4 \times \&c.}$, which Series in the Denominator must be continued to as many Terms as there are Units in m . Or if c stands for the Coefficient of the Term immediately preceding, then is $T = \frac{c}{m}z^m$.

Again, in the Series $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{30240}z^7 + \frac{1}{384000}z^9, \&c.$ (by which the Relation between the Circular Arch and its right Sine is express'd,) the Law of Continuation will be thus.

If

If T be any Term of the Series, whose order is express'd by m , and if c be the Coefficient immediately before ; then $T = \frac{-cz^{2m-1}}{2m-1 \times 2m-2}$.

And in the Reverse of this Series, or $z = y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{16128}y^9$, &c. the Law of Consecution will be thus. If T represents any Term, the Index of whose place in the Series is m , and if c be the preceding Coefficient ; then $T = \frac{2m-3 \times 2m-3 \times c}{2m-1 \times 2m-2} y^{2m-1}$. And the like of others.

44, 45, 46. If we would perform these Extractions after a more indefinite and general manner, we may proceed thus. Let the given Equation be $y^3 + a^2y + axy - 2a^3 - x^3 = 0$, the Terms of which should be disposed as in the Margin. Suppose $y = b + p$, where b is to be conceived as a near Approximation to the Root y , and p as its small Supplement. When this is substituted, the Equation will stand as it does here. Now because x and p are both small quantities, the most considerable quantities are at the beginning of the Equation, from whence they proceed gradually diminishing, both downwards and towards the right-hand ; as ought always to be suppos'd, when the Terms of an Equation are dispos'd according to a double Arithmetical Scale. And because instead of one unknown quantity y , we have here introduced two, b and p , we may determine one of them b , as the necessity of the Resolution shall require. To remove therefore the most considerable Quantities out of the Equation, and to leave only a Supplemental Equation, whose Root is p ; we may put $b^3 + a^2b - 2a^3 = 0$, which Equation will determine b , and which therefore henceforward we are to look upon as known. And for brevity sake, if we put $a^2 + 3b^2 = c$, we shall have the Equation in the Margin.

Now here, because the two initial Terms $+cp + abx$ are the most considerable of the Equation, which might be removed, if for the first Approximation to p we should assume $-\frac{abx}{c}$, and the resulting Supplemental Equation would be depress'd lower ; therefore make $p = -\frac{abx}{c} + q$, and by substitution we shall have this Equation following.

$$\left. \begin{array}{r} -2a^3 + a^2y + y^3 \\ * + axy * \\ * * * \\ -x^3 * * \end{array} \right\} = 0.$$

$$\left. \begin{array}{r} -2a^3 + a^2b + 3bp^2 + p^3 \\ + a^2p + 3b^2p \\ + abx + axp * \\ * * * \\ -x^3 * \end{array} \right\} = 0.$$

$$\left. \begin{array}{r} +cp + 3bp^2 + p^3 \\ + abx + axp * \\ * * * \\ -x^3 * \end{array} \right\} = 0.$$

Or

Or in this Equation, if we make $\frac{3a^2b^3}{c^2} - \frac{a^2b}{c} = d$, $a - \frac{6ab^2}{c} = e$, and $\frac{a^3b^3}{c^3} + 1 = f$; it will assume this form.

$$\left. \begin{array}{l} + \frac{cq}{c} xq \\ - \frac{6ab^2}{c} xq \\ + \frac{3bq^2}{c} xq^2 \\ + q^3 \end{array} \right\} - \frac{3ab}{c} xq^2 = 0.$$

$$\left. \begin{array}{l} + \frac{3a^2b^3}{c^2} x^2 \\ - \frac{a^2b}{c} x^2 \\ - \frac{a^3b^3}{c^3} x^3 \end{array} \right\} + \frac{3a^2b^2}{c^2} x^2q = 0.$$

$$\left. \begin{array}{l} - \frac{a^3b^3}{c^3} x^3 \\ - x^3 \end{array} \right\} = 0.$$

Here because the Terms to be next removed are $+cq + dx^2$, we may put $q = -\frac{d}{c}x^2 + r$, and by Substitution we shall have another Supplemental Equation, which will be farther depress'd, and so on as far as we please. Therefore we shall have the Root $y = b - \frac{ab}{c}x - \frac{d}{c}x^2$, &c. where b will be the Root of this Equation $b^3 + a^2b - 2a^3 = 0$, $c = a^2 + 3b^2$, $d = \frac{3a^2b^3}{c^2} - \frac{a^2b}{c}$, $e = a - \frac{6ab^2}{c}$, $f = \frac{a^3b^3}{c^3} + 1$, &c.

$$\left. \begin{array}{l} + cq \\ + exq \\ + dx^2 + \frac{3a^2b^2}{c^2} x^2q \\ - fx^3 \end{array} \right\} + \frac{3bq^2}{c} xq^2 + q^3 = 0.$$

Or by another Method of Solution, if in this Equation we assume (as before) $y = A + Bx + Cx^2 + Dx^3$, &c. and substitute this in the Equation, to determine the general Coefficients, we shall have $y = A - \frac{aA}{c^2}x + \frac{a^4A}{c^6}x^2 + \frac{c^8 + 7a^5A^3 - a^7A}{c^{10}}x^3$, &c. wherein A is the Root of the Equation $A^3 + a^2A - 2a^3 = 0$, and $c^2 = 3A^2 + a^2$.

47. All Equations cannot be thus immediately resolved, or their Roots cannot always be exhibited by an Arithmetical Scale, whose Root is one of the Quantities in the given Equation. But to perform the Analysis it is sometimes required, that a new Symbol or Quantity should be introduced into the Equation, by the Powers of which the Root to be extracted may be express'd in a converging Series. And the Relation between this new Symbol, and the Quantities of the Equation, must be exhibited by another Equation. Thus if it were proposed to extract the Root y of this Equation, $x = a + y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. it would be in vain to expect, that it might be express'd by the simple Powers of either x or a . For the Series itself supposes, in order to its converging, that y is some small Number less than Unity; but x and a are under no such limitations. And therefore a Series, composed of the ascending Powers of x , may be a diverging Series. It is therefore necessary to introduce a new Symbol, which shall also be small, that a Series

form'd of its Powers may converge to y . Now it is plain, that x and a , tho' ever so great, must always be near each other, because their difference $y - \frac{1}{2}y^2$, &c. is a small quantity. Assume therefore the Equation $x - a = z$, and z will be a small quantity as required; and being introduced instead of $x - a$, will give $z = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4$, &c. whose Root being extracted will be $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$, &c. as before.

48. Thus if we had the Equation $y^3 + y^2 + y - x^3 = 0$, to find the Root y ; we might have a Series for y composed of the ascending Powers of x , which would converge if x were a small quantity, less than Unity, but would diverge in contrary Circumstances. Supposing then that x was known to be a large Quantity; in this case the Author's Expedient is this. Making z the Reciprocal of x , or supposing the Equation $x = \frac{1}{z}$, instead of x he introduces z into the Equation, by which means he obtains a converging Series, consisting of the Powers of z ascending in the Numerators, that is in reality, of the Powers of x ascending in the Denominators. This he does, to keep within the Case he proposed to himself; but in the Method here pursued, there is no occasion to have recourse to this Expedient, it being an indifferent matter, whether the Powers of the converging quantity ascend in the Numerators or the Denominators.

Thus in the given Equation $y^3 + y^2 + y - x^3 = 0$, or (making $y = Ax^m$, &c.) $A^3x^{3m} + A^2x^{2m} + Ax^m - x^3 = 0$, &c.

by applying the Ruler we shall have the exterior Terms $A^3x^{3m} - x^3 = 0$, or $m = 1$, and $A = 1$. Also the resulting Number or Index is 3. The next Term to which the Ruler approaches will give $2m$, or 2; the last m , or 1. But 3, 2, 1, make a descending Progression, of which the common difference is 1. Therefore the form of the Root will be $y = Ax + B + Cx^{-1} + Dx^{-2}$, &c. which we may thus extract.

$$\begin{array}{l} y^3 \} = x^3 - x^2 - \frac{1}{3}x + \frac{2}{3}x^0 + \frac{7}{81}x^{-1}, \text{ \&c.} \\ + y^2 \} \quad \quad \quad + x^2 - \frac{2}{3}x - \frac{1}{3}x^0 + \frac{1}{81}x^{-1}, \text{ \&c.} \\ + y \} \quad \quad \quad + x - \frac{1}{3}x^0 - \frac{2}{9}x^{-1} + \frac{7}{81}x^{-2} + \frac{5}{81}x^{-3}, \text{ \&c.} \end{array}$$

Because $y^3 = x^3$, &c. it will be $y = x$, &c. and therefore $y^2 = x^2$, &c. which will make $y^3 = * - x^2$, &c. and (by Extraction) $y = * - \frac{1}{3}$, &c. Then (by squaring) $y^2 = * - \frac{2}{3}x$, &c. which with x below, and changing the Sign, makes $y^3 = * * - \frac{1}{3}x$, &c. and therefore

$y = ** - \frac{1}{9}x^{-1}$, &c. Then $y^2 = ** - \frac{1}{3}$, &c. and $y = * - \frac{1}{3}$, &c. which together, changing the Sign, make $y^3 = *** + \frac{2}{3}$, &c. and $y = *** + \frac{7}{81}x^{-2}$, &c. Then $y^2 = *** + \frac{1}{81}x^{-1}$, &c. and $y = ** - \frac{1}{9}x^{-1}$, &c. and therefore $y^3 = **** + \frac{7}{81}x^{-1}$, &c. and $y = **** + \frac{5}{81}x^{-2}$, &c.

Now as this Series is accommodated to the case of convergency when x is a large Quantity, so we may derive another Series from hence, which will be accommodated to the case when x is a small quantity. For the Ruler will direct us to the external Terms $Ax^m - x^3 = 0$, whence $m = 3$, and $A = 1$; and the resulting Number is 3. The next Term will give $2m$, or 6; and the last is $3m$, or 9. But 3, 6, 9 will form an ascending Progression, of which the common difference is 3. Therefore $y = Ax^3 + Bx^6 + Cx^9$, &c. will be the form of the Series in this case, which may be thus derived.

$$\begin{array}{l} y \} = x^3 - x^6 + x^9 \quad * - 4x^{12} + 14x^{18}, \text{ \&c.} \\ + y^2 \} - - - + x^6 - 2x^9 + 3x^{12} - 2x^{15} = 7x^{18}, \text{ \&c.} \\ + y^3 \} - - - - - + x^9 - 3x^{12} + 6x^{15} - 7x^{18}, \text{ \&c.} \end{array}$$

Here because $y = x^3$, &c. it will be $y^2 = x^6$, &c. and therefore $y = * - x^6$, &c. Then $y^2 = * - 2x^9$, &c. and $y^3 = x^9$, &c. and therefore $y = ** + x^9$, &c. Then $y^2 = ** + 3x^{12}$, &c. and $y^3 = * - 3x^{12}$, &c. and therefore $y = *** + 0$, &c.

The Expedient of the Ruler will indicate a third case of external Terms, which may be try'd also. For we may put $A^3x^{3m} + A^2x^{2m} + Ax^m = 0$, whence $m = 0$, and the Number resulting from the other Term is 3. Therefore 3 will be the common difference of the Progression, and the form of the Root will be $y = A + Bx^3 + Cx^6$, &c. But the Equation $A^3 + A^2 + A = 0$, will give $A = 0$, which will reduce this to the former Series. And the other two Roots of the Equation will be impossible.

If the Equation of this Example $y^3 + y^2 + y - x^3 = 0$ be multiply'd by the factor $y - 1$, we shall have the Equation $y^4 - y - x^3y + x^3 = 0$, or $y^4 * * - y - x^3y + x^3 \} = 0$, which when resolved, will only afford the same Series for the Root y as before.

49. This Equation $y^4 - x^2y^2 + xy^2 + 2y^2 - 2y + 1 = 0$, when reduced to the form of a double Arithmetical Scale, will stand as in the Margin.

Now the first Case of external Terms, shewn by the Ruler, in order for an ascending Series, will make $A^4x^{4m} + 2A^2x^{2m} - 2Ax^m + 1 = 0$, or $m = 0$; where the resulting Number is also 0. The second is $2m + 1$, or 1; the third $2m + 2$, or 2. Therefore the Arithmetical Progression will be 0, 1, 2, whose common difference is 1; and consequently it will be $y = A + Bx + Cx^2 + Dx^3, \&c.$ But the Equation $A^4 + 2A^2 - 2A + 1 = 0$, which should give the Value of the first Coefficient, will supply us with none but impossible Roots; so that y , the Root of this Equation, cannot be express'd by an Arithmetical Scale whose Root is x , or by an ascending Series that converges by the Powers of x , when x is a small quantity.

As for descending Series, there are two cases to be try'd; first the Ruler will give us $A^4x^{4m} - A^2x^{2m+2} = 0$, whence $4m = 2m + 2$, or $m = 1$, and $A = \pm 1$. The Number arising is 4; the next will be $2m + 1$, or 3; the next $2m$, or 2; the next m , or 1; the last 0. But the Arithmetical Progression 4, 3, 2, 1, 0, has 1 for its common difference, and therefore the form of the Series will be $y = Ax + B + Cx^{-1}, \&c.$ But to extract this Series by our usual Method, it will be best to reduce the Equation to this form, $y^2 - x^2 + x + 2 - 2y^{-1} + y^{-2} = 0$, and then to proceed thus:

$$\left. \begin{array}{l} y^2 \\ - 2y^{-1} \\ + y^{-2} \end{array} \right\} \begin{array}{l} = x^2 - x - 2 + 2x^{-1} - \frac{3}{2}x^{-2}, \&c. \\ \text{-----} \\ - 2x^{-1} + \frac{1}{2}x^{-2}, \&c. \\ \text{-----} \\ + x^{-2}, \&c. \end{array}$$

$$y = x - \frac{1}{2} - \frac{9}{8x} + \frac{7}{16x^2} - \frac{177}{128x^3}, \&c.$$

Because $y^2 = x^2 - x - 2, \&c.$ 'tis therefore (by Extraction) $y = x - \frac{1}{2} - \frac{9}{8}x^{-1}, \&c.$ Then (by Division) $-2y^{-1} = -2x^{-1}, \&c.$ so that $y^2 = * * * + 2x^{-1}, \&c.$ and (by Extraction) $y = * * * + \frac{7}{16}x^{-2}, \&c.$ Then $-2y^{-1} = * + \frac{1}{2}x^{-2}, \&c.$ and $y^{-2} = x^{-2}, \&c.$ which being united with a contrary sign, make $y^2 = * * * - \frac{3}{2}x^{-2}, \&c.$ and therefore by Extraction $y = * * * - \frac{1}{128}x^{-3}, \&c.$

In the other case of a descending Series we shall have the Equation $-A^2x^{2m+2} + 1 = 0$, whence $2m + 2 = 0$, or $m = -1$, and $A = \pm 1$. The Number hence arising is 0; the next will be $2m + 1$,

or -1 ; the next $2m$, or -2 ; and the last $4m$, or -4 . But the Numbers $0, -1, -2, -4$, will be found in a descending Arithmetical Progression, the common difference of which is 1 . Therefore the form of the Root is $y = Ax^{-1} + Bx^{-2} + Cx^{-3}, \&c.$ and the Terms of the Equation must be thus disposed for Resolution.

$$\begin{array}{rcl} y^{-2} & = & x^2 - x - 2 - \frac{5}{4}x^{-1} - \frac{13}{8}x^{-2}, \&c. \\ -2y^{-1} & + & 2x + 1 \\ + y^2 & - & 2x - 1 + \frac{5}{4}x^{-1} + \frac{5}{8}x^{-2}, \&c. \\ \hline y & = & \frac{1}{x} - \frac{1}{2x^2} + \frac{7}{8x^3} - \frac{7}{16x^4} + \frac{187}{128x^5}, \&c. \end{array}$$

Here because it is $y^{-2} = x^2, \&c.$ it will be by Extraction of the Square-root $y^{-1} = x, \&c.$ and by finding the Reciprocal, $y = x^{-1}, \&c.$ Then because $-2y^{-1} = -2x, \&c.$ this with a contrary Sign, and collected with $-x$ above, will make $y^{-2} = * + x, \&c.$ which (by Extraction) makes $y^{-1} = * + \frac{1}{2}, \&c.$ and by taking the Reciprocal, $y = * - \frac{1}{2}x^{-2}, \&c.$ Then because $-2y^{-1} = * - 1, \&c.$ this with a contrary sign, and collected with -2 above, will make $y^{-2} = * * - 1, \&c.$ and therefore (by Extraction) $y^{-1} = * * - \frac{5}{8}x^{-1}, \&c.$ and (by Division) $y = * * + \frac{7}{8}x^{-3}, \&c.$ Then because $-2y^{-1} = * * + \frac{5}{4}x^{-1}, \&c.$ it will be $y^{-2} = * * * - \frac{5}{4}x^{-1}, \&c.$ and $y^{-1} = * * * - \frac{5}{8}x^{-1}, \&c.$ and $y = * * * - \frac{7}{16}x^{-3}, \&c.$ Then because $-2y^{-1} = * * * + \frac{5}{8}x^{-2}, \&c.$ and $y^2 = x^{-2}, \&c.$ these collected with a contrary sign will make $y^{-2} = * * * * - \frac{13}{8}x^{-2}, \&c.$ and $y^{-1} = * * * * - \frac{5}{16}x^{-2}, \&c.$ and $y = * * * * + \frac{187}{128}x^{-4}, \&c.$

These are the two descending Series, which may be derived for the Root of this Equation, and which will converge by the Powers of x , when it is a large quantity. But if x should happen to be small, then in order to obtain a converging Series, we much change the Root of the Scale. As if it were known that x differs but little from 2 , we may conveniently put z for that small difference, or we may assume the Equation $x - 2 = z$. That is, instead of x in this Equation substitute $z + 2$, and we shall have a new Equation $y^4 - z^2y^2 - 3zy^2 - 2y + 1 = 0$, which will appear as in the Margin.

Here

Here to have an ascending Series, we must put $A + Az^m - 2Az^m + 1 = 0$, whence $m = 0$, and $A = 1$. The Number hence arising is 0; the next is $2m + 1$, or 1; and the last $2m + 2$, or 2.

But 0, 1, 2, are in an ascending Progression, whose common difference is 1. Therefore the form of the Series is $y = A + Bz + Cz^2 + Dz^3, \&c.$ And if the Root y be extracted by any of the foregoing Methods, it will be found $y = 1 + \frac{3}{2}z - \frac{7}{4}z^2, \&c.$ Also we may hence find two descending Series, which would converge by the Root of the Scale z , if it were a large quantity.

50, 51. Our Author has here opened a large field for the Solution of these Equations, by shewing, that the indeterminate quantity, or what we call the Root of the Scale, or the converging quantity, may be changed a great variety of ways, and thence new Series will be derived for the Root of the Equation, which in different circumstances will converge differently, so that the most commodious for the present occasion may always be chose. And when one Series does not sufficiently converge, we may be able to change it for another that shall converge faster. But that we may not be left to uncertain interpretations of the indeterminate quantity, or be obliged to make Suppositions at random; he gives us this Rule for finding initial Approximations, that may come at once pretty near the Root required, and therefore the Series will converge apace to it. Which Rule amounts to this: We are to find what quantities, when substituted for the indefinite Species in the proposed Equation, will make it divisible by the radical Species, increased or diminished by another quantity, or by the radical Species alone. The small difference that will be found between any one of those quantities, and the indeterminate quantity of the Equation, may be introduced instead of that indeterminate quantity, as a convenient Root of the Scale, by which the Series is to converge.

Thus if the Equation proposed be $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, and if for x we here substitute a , we shall have the Terms $y^3 + 2a^2y - 3a^3$, which are divisible by $y - a$, the Quotient being $y^2 + ay + 3a^2$. Therefore we may suppose, by the foregoing Rule, that $a - x = z$ is but a small quantity, or instead of x we may substitute $a - z$ in the proposed Equation, which will then become $y^3 + 2a^2y - azy + 3a^2z - 3az^2 + z^3 - 2a^3 = 0$. A Series

$$y^4 - 3z^2 - 2y + 1 \} = 0.$$

Or making $y = Az^m, \&c.$

$$A^4z^{4m} - 3A^2z^{2m+1} - 2Az^m + 1 \} = 0.$$

Series derived from hence, composed of the ascending Powers of z , must converge fast, *cæteris paribus*, because the Root of the Scale z is a small quantity.

Or in the same Equation, if for x we substitute $-a$, we shall have the Terms $y^3 - a^3$, which are divisible by $y - a$, the Quotient being $y^2 + ay + a^2$. Therefore we may suppose the difference between $-a$ and x to be but little, or that $-a - x = z$ is a small quantity, and therefore instead of x we may substitute its equal $-a - z$ in the given Equation. This will then become $y^3 - azy + 3a^2z + 3ax^2 - a^3 = 0$, where the Root y will converge by the Powers of the small quantity z .

Or if for x we substitute $-2a$, we shall have the Terms $y^3 - a^2y + 6a^3$, which are divisible by $y + 2a$, the Quotient being $y^2 - 2ay + 3a^2$. Wherefore we may suppose there is but a small difference between $-2a$ and x , or that $-2a - x = z$ is a small quantity; and therefore instead of x we may introduce its equal $-2a - z$ into the Equation, which will then become $y^3 - a^2y - azy + 6a^3 + 12a^2z + 6az^2 + z^3 = 0$.

Lastly, if for x we substitute $-2^{\frac{1}{3}}a$, we shall have the Terms $y^3 - 2^{\frac{1}{3}}a^2y + a^2y$, which are divisible by y , the Radical Species alone. Wherefore we may suppose there is but a small difference between $-2^{\frac{1}{3}}a$ and x , or that $-2^{\frac{1}{3}}a - x = z$ is a small quantity; and therefore instead of x we may substitute its equal $-2^{\frac{1}{3}}a - z$, which will reduce the Equation to $y^3 + 1 - \sqrt[3]{2} \times a^2y - azy + 3\sqrt[3]{4} \times a^2z + 3\sqrt[3]{2} \times az^2 + z^3 = 0$, wherein the Series for the Root y may converge by the Powers of the small quantity z .

But the reason of this Operation still remains to be inquired into, which I shall endeavour to explain from the present Example. In the Equation $y^3 + azy + a^2y - x^3 - 2a^3 = 0$, the indeterminate quantity x , of its own nature, must be susceptible of all possible Values; at least, if it had any limitations, they would be shew'd by impossible Roots. Among other values, it will receive these, a , $-a$, $-2a$, $-2^{\frac{1}{3}}a$, &c. in which cases the Equation would become $y^3 + 2a^2y - 3a^3 = 0$, $y^3 - a^3 = 0$, $y^3 - a^2y + 6a^3 = 0$, $y^3 - 2^{\frac{1}{3}}a^2y + a^2y = 0$, &c. respectively. Now as these Equations admit of just Roots, as appears by their being divisible by $y +$ or $-$ another quantity, and the last by y alone; so that in the Resolution, the whole Equation (in those cases) would be immediately exhausted: And in other cases, when x does not much recede from one of those Values,

Values, the Equation would be nearly exhausted. Therefore the introducing of z , which is the small difference between x and any one of those Values, must depress the Equation; and z itself must be a convenient quantity to be made the Root of the Scale, or the converging Quantity.

I shall give the Solution of one of the Equations of these Examples, which shall be this, $y^3 - azy + 3a^2z + 3az^2 - a^3 = 0$, or

$$\left. \begin{array}{l} y^3 * * - a^3 \\ - azy + 3a^2z \\ + 3az^2 \end{array} \right\} = 0. \quad \left. \begin{array}{l} y^3 \\ - azy \end{array} \right\} = \left. \begin{array}{l} a^3 - 3a^2z - 3az^2 \\ + a^2z - \frac{2}{3}az^2 - \frac{5}{3}z^3, \&c. \\ - a^2z + \frac{2}{3}az^2 + \frac{5}{3}z^3, \&c. \end{array} \right\}$$

$$y = a - \frac{1}{3}z - \frac{5z^2}{3a} - \frac{217z^3}{81a^2}, \&c.$$

Here because $y^3 = a^3$, &c. it will be $y = a$, &c. Then $-azy = -a^2z$, &c. which must be wrote again with a contrary sign, and united with $-3a^2z$ above, to make $y^3 = * - 2a^2z$, &c. and therefore $y = * - \frac{2}{3}z$, &c. Then $-azy = * + \frac{2}{3}az^2$, &c. and $y^3 = * * - \frac{1}{3}az^2$, &c. and $y = * * - \frac{5z^2}{3a}$, &c. Then $-azy = * * + \frac{5}{3}z^3$, &c. and $y^3 = * * * - \frac{5}{3}z^3$, &c. and $y = * * * - \frac{217z^3}{81a^2}$, &c.

The Author hints at many other ways of deriving a variety of Series from the same Equation; as when we suppose the afore-mentioned difference z to be indefinitely great, and from that Supposition we find Series, in which the Powers of z shall ascend in the Denominators. This Case we have all along pursued indiscriminately with the other Case, in which the Powers of the converging quantity ascend in the Numerators, and therefore we need add nothing here about it. Another Expedient is, to assume for the converging quantity some other quantity of the Equation, which then may be consider'd as indeterminate. So here, for instance, we may change a into x , and x into a . Or lastly, to assume any Relation at pleasure, (suppose $x = az + bz^2$, $x = \frac{a}{b+z}$, $x = \frac{a+cz}{b+z}$, &c.) between the indeterminate quantity of the Equation x , and the quantity z we would introduce into its room, by which new equivalent Equations may be form'd, and then their Roots may be extracted. And afterwards the value of z may be express'd by x , by means of the assumed Equation.

52. The Author here, in a summary way, gives us a *Rationale* of his whole Method of Extractions, proving *à priori*, that the Series thus form'd, and continued *in infinitum*, will then be the just Roots of the proposed Equation. And if they are only continued to a competent number of Terms, (the more the better,) yet then will they be a very near Approximation to the just and compleat Roots. For, when an Equation is proposed to be resolved, as near an Approach is made to the Root, suppose y , as can be had in a single Term, compos'd of the quantities given by the Equation; and because there is a Remainder, a Residual or Secondary Equation is thence form'd, whose Root p is the Supplement to the Root of the given Equation, whatever that may be. Then as near an approach is made to p , as can be done by a single Term, and a new Residual Equation is form'd from the Remainder, wherein the Root q is the Supplement to p . And by proceeding thus, the Residual Equations are continually depress'd, and the Supplements grow perpetually less and less, till the Terms at last are less than any assignable quantities. We may illustrate this by a familiar Example, taken from the usual Method of Division of Decimal Fractions. At every Operation we put as large a Figure in the Quotient, as the Dividend and Divisor will permit, so as to leave the least Remainder possible. Then this Remainder supplies the place of a new Dividend, which we are to exhaust as far as can be done by one Figure, and therefore we put the greatest number we can for the next Figure of the Quotient, and thereby leave the least Remainder we can. And so we go on, either till the whole Dividend is exhausted, if that can be done, or till we have obtain'd a sufficient Approximation in decimal places or figures. And the same way of Argumentation, that proves our Author's Method of Extraction, may easily be apply'd to the other ways of Analysis that are here found.

53, 54. Here it is seasonably observed, that tho' the indefinite Quantity should not be taken so small, as to make the Series converge very fast, yet it would however converge to the true Root, tho' by more steps and slower degrees. And this would obtain in proportion, even if it were taken never so large, provided we do not exceed the due Limits of the Roots, which may be discover'd, either from the given Equation, or from the Root when exhibited by a Series, or may be farther deduced and illustrated by some Geometrical Figure, to which the Equation is accommodated.

So if the given Equation were $yy = ax - xx$, it is easy to observe, that neither y nor x can be infinite, but they are both liable to

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several

several Limitations. For if x be suppos'd infinite, the Term ax would vanish in respect of $-xx$, which would give the Value of yy impossible on this Supposition. Nor can x be negative; for then the Value of yy would be negative, and therefore the Value of y would again become impossible. If $x = 0$, then is $y = 0$ also; which is one Limitation of both quantities. As yy is the difference between ax and xx , when that difference is greatest, then will yy , and consequently y , be greatest also. But this happens when $x = \frac{1}{2}a$, as also $y = \frac{1}{2}a$, as may appear from the following Prob. 3. And in general, when y is express'd by any number of Terms, whether finite or infinite, it will then come to its Limit when the difference is greatest between the affirmative and negative Terms; as may appear from the same Problem. This last will be a Limitation for y , but not for x . Lastly, when $x = a$, then $y = 0$; which will limit both x and y . For if we suppose x to be greater than a , the negative Term will prevail over the affirmative, and give the Value of yy negative, which will make the Value of y impossible. So that upon the whole, the Limitations of x in this Equation will be these, that it cannot be less than 0, nor greater than a , but may be of any intermediate magnitude between those Limits.

Now if we resolve this Equation, and find the Value of y in an infinite Series, we may still discover the same Limitations from thence. For from the Equation $yy = ax - xx$, by extracting the

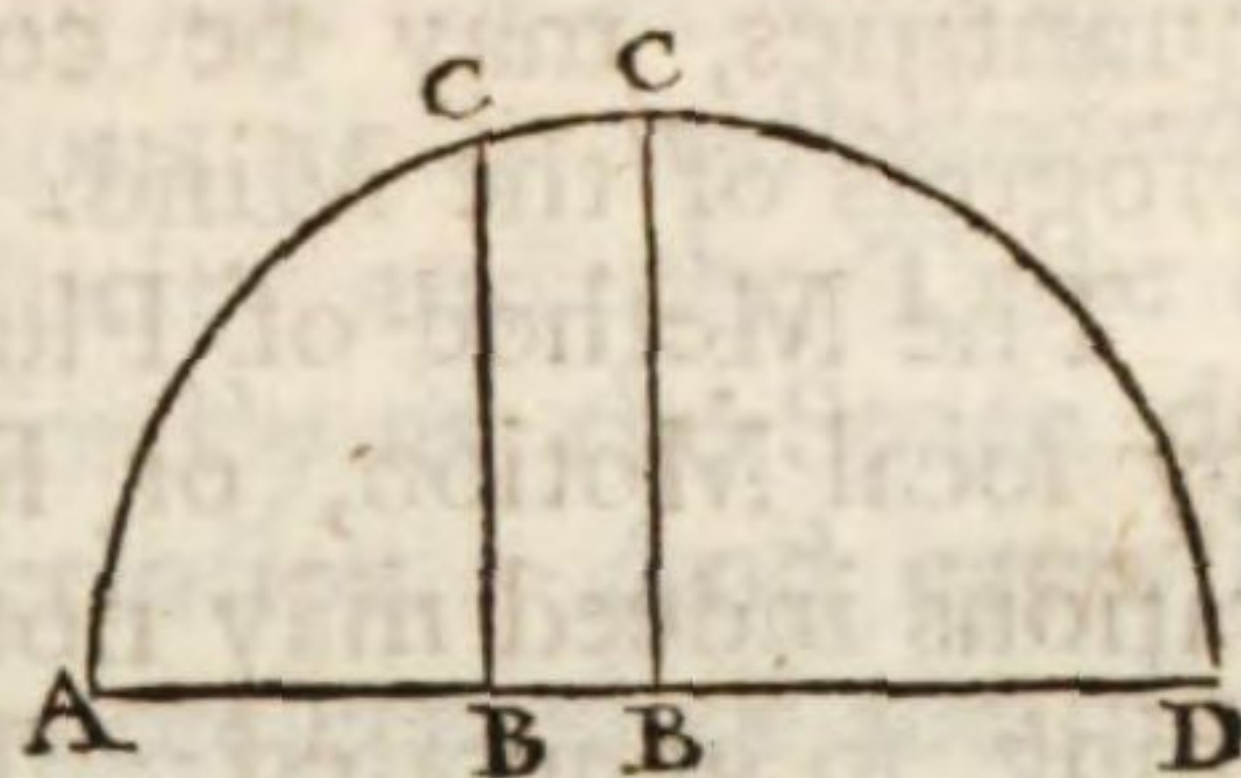
square-root, as before, we shall have $y = a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16a^{\frac{5}{2}}}$, c. that is, $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$ into $1 - \frac{x}{2a} - \frac{x^2}{8a^2} - \frac{x^3}{16a^3}$, &c. Here

x cannot be negative; for then $x^{\frac{1}{2}}$ would be an impossible quantity. Nor can x be greater than a ; for then the converging quantity $\frac{x}{a}$, or the Root of the Scale by which the Series is express'd, would be greater than Unity, and consequently the Series would diverge, and not converge as it ought to do. The Limit between converging and diverging will be found, by putting $x = a$, and therefore $y = 0$; in which case we shall have the identical Numeral Series $1 = \frac{1}{2} + \frac{1}{8} + \frac{1}{16}$, &c. of the same nature with some of those, which we have elsewhere taken notice of. So that we may take x of any intermediate Value between 0 and a , in order to have a converging Series. But the nearer it is taken to the Limit 0, so much faster the Series will converge to the true Root; and the nearer it is taken to the Limit a , it will converge so much the slower. But it will
however

however converge, if x be taken never so little less than a . And by Analogy, a like Judgment is to be made in all other cases.

The Limits and other affections of y are likewise discoverable from this Series. When $x = 0$, then $y = 0$. When x is a nascent quantity, or but just beginning to be positive, all the Terms but the first may be neglected, and y will be a mean proportional between a and x . Also $y = 0$, when the affirmative Term is equal to all the negative Terms, or when $1 = \frac{x}{2a} + \frac{x^2}{8a^2} + \frac{x^3}{16a^3}$, &c. that is, when $x = a$. For then $1 = \frac{1}{2} + \frac{1}{8} + \frac{1}{8}$, &c. as above. Lastly, y will be a *Maximum* when the difference between the affirmative Term and all the negative Terms is greatest, which by Prob. 3. will be found when $x = \frac{1}{2}a$.

Now the Figure or Curve that may be adapted to this Equation, and to this Series, and which will have the same Limitations that they have, is the Circle ACD, whose Diameter is $AD = a$, its Absciss $AB = x$, and its perpendicular Ordinate $BC = y$. For as the Ordinate $BC = y$ is a mean proportional between the Segments of the Diameter $AB = x$ and $BD = a - x$, it will be $yy = ax - xx$. And therefore the Ordinate $BC = y$ will be express'd by the foregoing Series. But it is plain from the nature of the Circle, that the Absciss AB cannot be extended backwards, so as to become negative; neither can it be continued forwards beyond the end of the Diameter D . And that at A and D , where the Diameter begins and ends, the Ordinate is nothing. And the greatest Ordinate is at the Center, or when $AB = \frac{1}{2}AD$.



SECT. VI. Transition to the Method of Fluxions.

55. **T**HE learned and sagacious Author having thus accomplish'd one part of his design, which was, to teach the Method of converting all kinds of Algebraic Quantities into simple Terms, by reducing them to infinite Series: He now goes on to shew the use and application of this Reduction, or of these Series, in the Method of Fluxions, which is indeed the principal design of this Treatise. For this Method has so near a connexion with, and dependence upon the foregoing, that it would be very lame and defective without it. He lays down the fundamental Principles of

this Method in a very general and scientifick manner, deducing them from the received and known laws of local Motion. Nor is this inverting the natural order of Science, as some have pretended, by introducing the Doctrine of Motion into pure Geometrical Speculations. For Geometrical and Analytical Quantities are best conceived as generated by local Motion; and their properties may as well be derived from them while they are generating, as when their generation is suppos'd to be already accomplish'd, in any other way. A right line, or a curve line, is described by the motion of a point, a surface by the motion of a line, a solid by the motion of a surface, an angle by the rotation of a radius; all which motions we may conceive to be perform'd according to any stated law, as occasion shall require. These generations of quantities we daily see to obtain *in rerum naturâ*, and is the manner the ancient Geometricians had often recourse to, in considering their production, and then deducing their properties from such actual descriptions. And by analogy, all other quantities, as well as these continued geometrical quantities, may be conceived as generated by a kind of motion or progress of the Mind.

The Method of Fluxions then supposes quantities to be generated by local Motion, or something analogous thereto, tho' such generations indeed may not be essentially necessary to the nature of the thing so generated. They might have an existence independent of these motions, and may be conceived as produced many other ways, and yet will be endued with the same properties. But this conception, of their being now generated by local Motion, is a very fertile notion, and an exceeding useful artifice for discovering their properties, and a great help to the Mind for a clear, distinct, and methodical perception of them. For local Motion supposes a notion of time, and time implies a succession of Ideas. We easily distinguish it into what was, what is, and what will be, in these generations of quantities; and so we commodiously consider those things by parts, which would be too much for our faculties, and extremely difficult for the Mind to take in the whole together, without such artificial partitions and distributions.

Our Author therefore makes this easy supposition, that a Line may be conceived as now describing by a Point, which moves either equably or inequably, either with an uniform motion, or else according to any rate of continual Acceleration or Retardation. Velocity is a Mathematical Quantity, and like all such, it is susceptible of infinite gradations, may be intended or remitted, may be increased

or

or diminish'd in different parts of the space described, according to an infinite variety of stated Laws. Now it is plain, that the space thus described, and the law of acceleration or retardation, (that is, the velocity at every point of time,) must have a mutual relation to each other, and must mutually determine each other; so that one of them being assign'd, the other by necessary inference may be derived from it. And therefore this is strictly a Geometrical Problem, and capable of a full Determination. And all Geometrical Propositions once demonstrated, or duly investigated, may be safely made use of, to derive other Propositions from them. This will divide the present Problem into two Cases, according as either the Space or Velocity is assign'd, at any given time, in order to find the other. And this has given occasion to that distinction which has since obtain'd, of the *direct and inverse Method of Fluxions*, each of which we shall now consider apart.

56. In the direct Method the Problem is thus abstractedly proposed. *From the Space described, being continually given, or assumed, or being known at any point of Time assign'd; to find the Velocity of the Motion at that Time.* Now in equable Motions it is well known, that the Space described is always as the Velocity and the Time of description conjunctly; or the Velocity is directly as the Space described, and reciprocally as the Time of description. And even in inequable Motions, or such as are continually accelerated or retarded, according to some stated Law, if we take the Spaces and Times very small, they will make a near approach to the nature of equable Motions; and still the nearer, the smaller those are taken. But if we may suppose the Times and Spaces to be indefinitely small, or if they are nascent or evanescent quantities, then we shall have the Velocity in any infinitely little Space, as that Space directly, and as the *tempusculum* inversely. This property therefore of all inequable Motions being thus deduced, will afford us a medium for solving the present Problem, as will be shewn afterwards. So that the Space described being thus continually given, and the whole time of its description, the Velocity at the end of that time will be thence determinable.

57. The general abstract Mechanical Problem, which amounts to the same as what is call'd the inverse Method of Fluxions, will be this. *From the Velocity of the Motion being continually given, to determine the Space described, at any point of Time assign'd.* For the Solution of which we shall have the assistance of this Mechanical Theorem, that in inequable Motions, or when a Point describes a
Line

Line according to any rate of acceleration or retardation, the indefinitely little Space described in any indefinitely little Time, will be in a compound ratio of the Time and the Velocity; or the *spatiolum* will be as the velocity and the *tempusculum* conjunctly. This being the Law of all equable Motions, when the Space and Time are any finite quantities, it will obtain also in all inequable Motions, when the Space and Time are diminish'd *in infinitum*. For by this means all inequable Motions are reduced, as it were, to equability. Hence the Time and the Velocity being continually known, the Space described may be known also; as will more fully appear from what follows. This Problem, in all its cases, will be capable of a just determination; tho' taking it in its full extent, we must acknowledge it to be a very difficult and operose Problem. So that our Author had good reason for calling it *molestissimum & omnium difficillimum problema*.

58. To fix the Ideas of his Reader, our Author illustrates his general Problems by a particular Example. If two Spaces x and y are described by two points in such manner, that the Space x being uniformly increased, in the nature of Time, and its equable velocity being represented by the Symbol $\dot{x}$; and if the Space y increases inequably, but after such a rate, as that the Equation $y = xx$ shall always determine the relation between those Spaces; (or x being continually given, y will be thence known;) then the velocity of the increase of y shall always be represented by $2x\dot{x}$. That is, if the symbol $\dot{y}$ be put to represent the velocity of the increase of y , then will the Equation $\dot{y} = 2x\dot{x}$ always obtain, as will be shewn hereafter. Now from the given Equation $y = xx$, or from the relation of the Spaces y and x , (that is, the Space and Time, or its representative,) being continually given, the relation of the Velocities $\dot{y} = 2x\dot{x}$ is found, or the relation of the Velocity $\dot{y}$, by which the Space increases, to the Velocity $\dot{x}$, by which the representative of the Time increases. And this is an instance of the Solution of the first general Problem, or of a particular Question in the direct Method of Fluxions. But *vice versa*, if the last Equation $\dot{y} = 2x\dot{x}$ were given, or if the Velocity $\dot{y}$, by which the Space y is described, were continually known from the Time x being given, and its Velocity $\dot{x}$; and if from thence we should derive the Equation $y = xx$, or the relation of the Space and Time: This would be an instance of the Solution of the second general Problem, or of a particular Question of the inverse Method of Fluxions. And in analogy to this description of Spaces by moving points, our Author considers all other quantities whatever as generated

nerated and produced by continual augmentation, or by the perpetual accession and accretion of new particles of the same kind.

59. In settling the Laws of his Calculus of Fluxions, our Author very skilfully and judiciously disengages himself from all consideration of Time, as being a thing of too Physical or Metaphysical a nature to be admitted here, especially when there was no absolute necessity for it. For tho' all Motions, and Velocities of Motion, when they come to be compared or measured, may seem necessarily to include a notion of Time; yet Time, like all other quantities, may be represented by Lines and Symbols, as in the foregoing example, especially when we conceive them to increase uniformly. And these representatives or proxies of Time, which in some measure may be made the objects of Sense, will answer the present purpose as well as the thing itself. So that Time, in some sense, may be said to be eliminated and excluded out of the inquiry. By this means the Problem is no longer Physical, but becomes much more simple and Geometrical, as being wholly confined to the description of Lines and Spaces, with their comparative Velocities of increase and decrease. Now from the equable Flux of Time, which we conceive to be generated by the continual accession of new particles, or Moments, our Author has thought fit to call his Calculus *the Method of Fluxions*.

60, 61. Here the Author premises some Definitions, and other necessary preliminaries to his Method. Thus Quantities, which in any Problem or Equation are suppos'd to be susceptible of continual increase or decrease, he calls *Fluents*, or *flowing Quantities*; which are sometimes call'd *variable* or *indeterminate quantities*, because they are capable of receiving an infinite number of particular values, in a regular order of succession. The Velocities of the increase or decrease of such quantities are call'd their *Fluxions*; and quantities in the same Problem, not liable to increase or decrease, or whose Fluxions are nothing, are call'd *constant*, *given*, *invariable*, and *determinate quantities*. This distinction of quantities, when once made, is carefully observed through the whole Problem, and insinuated by proper Symbols. For the first Letters of the Alphabet are generally appropriated for denoting constant quantities, and the last Letters commonly signify variable quantities, and the same Letters, being pointed, represent the Fluxions of those variable quantities or Fluents respectively. This distinction between these quantities is not altogether arbitrary, but has some foundation in the nature of the thing, at least during the Solution of the present Problem. For the flowing
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or variable quantities may be conceived as *now generating* by Motion, and the constant or invariable quantities as some how or other *already generated*. Thus in any given Circle or Parabola, the Diameter or Parameter are constant lines, or already generated; but the Absciss, Ordinate, Area, Curve-line, &c. are flowing and variable quantities, because they are to be understood as now describing by local Motion, while their properties are derived. Another distinction of these quantities may be this. A constant or given line in any Problem is *linea quædam*, but an indeterminate line is *linea quævis vel quæcunque*, because it may admit of infinite values. Or lastly, constant quantities in a Problem are those, whose ratio to a common Unit, of their own kind, is suppos'd to be known; but in variable quantities that ratio cannot be known, because it is varying perpetually. This distinction of quantities however, into determinate and indeterminate, subsists no longer than the present Calculation requires; for as it is a distinction form'd by the Imagination only, for its own conveniency, it has a power of abolishing it, and of converting determinate quantities into indeterminate, and *vice versâ*, as occasion may require; of which we shall see Instances in what follows. In a Problem, or Equation, there may be any number of constant quantities, but there must be at least two that are flowing and indeterminate; for one cannot increase or diminish, while all the rest continue the same. If there are more than two variable quantities in a Problem, their relation ought to be exhibited by more than one Equation.





ANNOTATIONS on Prob. I.

O R,

The relation of the flowing Quantities being given,
to determine the relation of their Fluxions.

SECT. I. *Concerning Fluxions of the first order, and to
find their Equations.*

I. **T**HE Author having thus proposed his fundamental Problems, in an abstract and general manner, and gradually brought them down to the form most convenient for his Method; he now proceeds to deliver the Precepts of Solution, which he illustrates by a sufficient variety of Examples; reserving the Demonstration to be given afterwards, when his Readers will be better prepared to apprehend the force of it, and when their notions will be better settled and confirm'd. These Precepts of Solution, or the Rules for finding the Fluxions of any given Equation, are very short, elegant, and comprehensive; and appear to have but little affinity with the Rules usually given for this purpose: But that is owing to their great degree of universality. We are to form, as it were, so many different Tables for the Equation, as there are flowing quantities in it, by disposing the Terms according to the Powers of each quantity, so as that their Indices may form an Arithmetical Progression. Then the Terms are to be multiply'd in each case, either by the Progression of the Indices, or by the Terms of any other Arithmetical Progression, (which yet should have the same common difference with the Progression of the Indices;)

as also by the Fluxion of that Fluent, and then to be divided by the Fluent itself. Last of all, these Terms are to be collected, according to their proper Signs, and to be made equal to nothing; which will be a new Equation, exhibiting the relation of the Fluxions. This process indeed is not so short as the Method for taking Fluxions, (to be given presently,) which he elsewhere delivers, and which is commonly follow'd; but it makes sufficient amends by the universality of it, and by the great variety of Solutions which it will afford. For we may derive as many different Fluxional Equations from the same given Equation, as we shall think fit to assume different Arithmetical Progressions. Yet all these Equations will agree in the main, and tho' differing in form, yet each will truly give the relation of the Fluxions, as will appear from the following Examples.

2. In the first Example we are to take the Fluxions of the Equation $x^3 - ax^2 + axy - y^3 = 0$, where the Terms are always brought over to one side. These Terms being disposed according to the powers of the Fluent x , or being consider'd as a Number express'd by the Scale whose Root is x , will stand thus $x^3 - ax^2 + ayx^1 - y^3x^0 = 0$; and assuming the Arithmetical Progression 3, 2, 1, 0, which is here that of the Indices of x , and multiplying each Term by each respectively, we shall have the Terms $3x^3 - 2ax^2 + ayx^*$; which again multiply'd by $\frac{\dot{x}}{x}$, or $\dot{x}x^{-1}$, according to the Rule, will make $3\dot{x}x^2 - 2a\dot{x}x + ay\dot{x}$. Then in the same Equation making the other Fluent y the Root of the Scale, it will stand thus, $-y^3 + 0y^2 + axy^1 - ax^2y^0 = 0$; and assuming the Arith-

metical Progression 3, 2, 1, 0, which also is the Progression of the Indices of y , and multiplying as before, we shall have the Terms $-3y^3 + axy^*$, which multiply'd by $\frac{\dot{y}}{y}$, or $\dot{y}y^{-1}$, will make $-3\dot{y}y^2 + ax\dot{y}$. Then collecting the Terms, the Equation $3\dot{x}x^2 - 2a\dot{x}x + ay\dot{x} - 3\dot{y}y^2 + ax\dot{y} = 0$ will give the required relation of the Fluxions. For if we resolve this Equation into an Analogy, we shall have $\dot{x} : \dot{y} :: 3y^2 - ax : 3x^2 - 2ax + ay$; which, in all the values that x and y can assume, will give the ratio of their Fluxions, or the comparative velocity of their increase or decrease, when they flow according to the given Equation.

Or to find this ratio of the Fluxions more immediately, or the value of the Fraction $\frac{\dot{y}}{\dot{x}}$ by fewer steps, we may proceed thus. Write down the Fraction $\frac{\dot{y}}{\dot{x}}$ with the note of equality after it, and in the

Numerator

Numerator of the equivalent Fraction write the Terms of the Equation, dispos'd according to x , with their respective signs; each being multiply'd by the Index of x in that Term, (increas'd or diminish'd, if you please, by any common Number,) as also divided by x . In the Denominator do the same by the Terms, when dispos'd according to y , only changing the signs. Thus in the present Equation $x^3 - ax^2 + axy - y^3 = 0$, we shall have at once

$$\frac{\dot{y}}{\dot{x}} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}.$$

Let us now apply the Solution another way. The Equation $x^3 - ax^2 + axy - y^3 = 0$ being order'd according to x as before, will be $x^3 - ax^2 + ayx^1 - y^3x^0 = 0$; and supposing the Indices of x to be increas'd by an unit, or assuming the Arithmetical Progression $\frac{4x}{x}, \frac{3x}{x}, \frac{2x}{x}, \frac{x}{x}$, and multiplying the Terms respectively, we shall have these Terms $4\dot{x}x^2 - 3a\dot{x}x + 2ay\dot{x} - y^3\dot{x}x^{-1}$. Then ordering the Terms according to y , they will become $-y^3 + 0y^2 + axy^1 + x^3y^0 = 0$; and supposing the Indices of y to be diminish'd

$$-ax^2$$

by an unit, or assuming the Arithmetical Progression $\frac{2y}{y}, \frac{y}{y}, \frac{0y}{y}, \frac{-y}{y}$, and multiplying the Terms respectively, we shall have these Terms $-2\dot{y}y^2 - x^3\dot{y}y^{-1} + ax^2\dot{y}y^{-1}$. So that collecting the Terms, we shall have $4\dot{x}x^2 - 3a\dot{x}x + 2ay\dot{x} - y^3\dot{x}x^{-1} - 2\dot{y}y^2 - x^3\dot{y}y^{-1} + ax^2\dot{y}y^{-1} = 0$, for the Fluxional Equation required. Or the ratio of the Fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{4x^2 - 3ax + 2ay - y^3x^{-1}}{2y^2 - x^3y^{-1} - ax^2y^{-1}}$; which ratio may be found immediately by applying the foregoing Rule.

Or contrary-wise, if we multiply the Equation in the first form by the Progression $\frac{2x}{x}, \frac{x}{x}, \frac{0x}{x}, \frac{-x}{x}$, we shall have the Terms $2\dot{x}x^2 - a\dot{x}x + y^3\dot{x}x^{-1}$. And if we multiply the Equation in the second form by $\frac{4y}{y}, \frac{3y}{y}, \frac{2y}{y}, \frac{y}{y}$, we shall have the Terms $-4\dot{y}y^2 + 2ax\dot{y} + x^3\dot{y}y^{-1} - ax^2\dot{y}y^{-1}$. Therefore collecting 'tis $2\dot{x}x^2 - a\dot{x}x + y^3\dot{x}x^{-1} - 4\dot{y}y^2 + 2ax\dot{y} + x^3\dot{y}y^{-1} - ax^2\dot{y}y^{-1} = 0$. Or the ratio of the Fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{2x^2 - ax + y^3x^{-1}}{4y^2 - 2ax - x^3y^{-1} + ax^2y^{-1}}$, which might have been found at once by the foregoing Rule.

And in general, if the Equation $x^3 - ax^2 + axy - y^3 = 0$, in the form $x^3 - ax^2 + axy - y^3x^0 = 0$, be multiply'd by the Terms of this Arithmetical Progression $\frac{m+3}{x}\dot{x}, \frac{m+2}{x}\dot{x}, \frac{m+1}{x}\dot{x}, \frac{m}{x}\dot{x}$; it will produce the Terms $\overline{m+3}\dot{x}x^2 - \overline{m+2}a\dot{x}x + \overline{m+1}ax\dot{y} - m\dot{y}y^3x^{-1}$;

and if the same Equation, reduced to the form $-y^3 + 3xy^2 + ax^2y - x^3y^0 = 0$, be multiply'd by the Terms of this Arithmetical Progression $\frac{n+3}{y}y, \frac{n+2}{y}y, \frac{n+1}{y}y, \frac{n}{y}y$, it will produce the Terms $-n+3yy^2$

$* + n + 1axy + nx^2yy^{-1} - nax^2yy^{-1}$. Then collecting the Terms, we shall have $m + 3xx^2 - m + 2axx + m + 1axy - my^3xx^{-1} - n + 3yy^2 * + n + 1axy + nx^2yy^{-1} - nax^2yy^{-1} = 0$, for the Fluxional Equation required. Or the ratio of the Fluxions will be

$\frac{y}{x} = \frac{m + 3x^2 - m + 2ax + m + 1ay - my^3x^{-1}}{n + 3y^2 * - n + 1ax - nx^2y^{-1} + nax^2y^{-1}}$; which might have been found immediately from the given Equation, by the foregoing Rule.

Here the general Numbers m and n may be determined *pro lubitu*, by which means we may obtain as many Fluxional Equations as we please, which will all belong to the given Equation. And thus we may always find the simplest Expression, or that which is best accommodated to the present exigence. Thus if we make $m = 0$, and $n = 0$, we shall have $\frac{y}{x} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}$, as found before. Or if

we make $m = 1$, and $n = -1$, we shall have $\frac{y}{x} = \frac{4x^2 - 3ax + 2ay - y^3x^{-1}}{2y^2 + x^2y^{-1} - ax^2y^{-1}}$, as before. Or if we make $m = -1$, and $n = 1$, we shall have

$\frac{y}{x} = \frac{2x^2 - ax + y^3x^{-1}}{4y^2 - 2ax - x^2y^{-1} + ax^2y^{-1}}$, as before. Or if we make $m = -3$, and $n = -3$, we shall have

$\frac{y}{x} = \frac{* ax - 2ay + 3y^3x^{-1}}{* * 2ax + 3x^2y^{-1} - 3ax^2y^{-1}} = \frac{ax^2y - 2axy^2 + 3y^4}{2ax^2y + 3x^4 - 3ax^3}$. And so of others. Now this variety of Solutions

will beget no ambiguity in the Conclusion, as possibly might have been suspected; for it is no other than what ought necessarily to arise, from the different forms the given Equation may acquire, as will appear afterwards. If we confine ourselves to the Progression of the Indices, it will bring the Solution to the common Method of taking Fluxions, which our Author has taught elsewhere, and which, because it is easy and expeditious, and requires no certain order of the Terms, I shall here subjoin.

For every Term of the given Equation, so many Terms must be form'd in the Fluxional Equation, as there are flowing Quantities in that Term. And this must be done, (1.) by multiplying the Term by the Index of each flowing Quantity contain'd in it. (2.) By dividing it by the quantity itself; and, (3.) by multiplying by its Fluxion. Thus in the foregoing Equation $x^3 - ax^2 + axy - y^3 = 0$, the Fluxion belonging to the Term x^3 is $\frac{3x^2x}{x}$, or $3x^2x$.

The

The Fluxion belonging to $-ax^2$ is $-\frac{2ax\dot{x}}{x}$, or $-2ax\dot{x}$. The Fluxion belonging to ayx is $\frac{ayx\dot{y}}{y} + \frac{ayx\dot{x}}{x}$, or $axy\dot{y} + ay\dot{x}$. And the Fluxion belonging to $-y^3$ is $-\frac{3y^2\dot{y}}{y}$, or $-3y^2\dot{y}$. So that the Fluxion of the whole Equation, or the whole Fluxional Equation, is $3x^2\dot{x} - 2ax\dot{x} + ayx\dot{y} + ay\dot{x} - 3y^2\dot{y} = 0$. Thus the Equation $x^m = y$, will give $mxx^{m-1} = \dot{y}$; and the Equation $x^m z^n = y$, will give $mxx^{m-1}z^n + nx^m z^{n-1} = \dot{y}$ for its Fluxional Equation. And the like of other Examples.

If we take the Author's simple Example, in pag. 19, or the Equation $y = ax$, or rather $ay - x^2 = 0$, that is $ayx^0 - x^2y^0 = 0$, in order to find its most general Fluxional Equation; it may be perform'd by the Rule before given, supposing the Index of x to be increas'd by m , and the Index of y by n . For then we shall have directly $\frac{\dot{y}}{\dot{x}} = \frac{mayx^{-1} - \overline{m+2x}}{nx^2y^{-1} - \overline{n+1a}}$. For the first Term of the given Equation being ayx^0 , this multiply'd by the Index of x increas'd by m , that is by m , and divided by x , will give $mayx^{-1}$ for the first Term of the Numerator. Also the second Term being $-x^2y^0$, this multiply'd by the Index of x increas'd by m , that is by $m+2$, and divided by x , will give $-\overline{m+2x}$ for the second Term of the Numerator. Again, the first Term of the given Equation may be now $-x^2y^0$, which multiply'd by the Index of y increas'd by n , that is by n , and divided by y , will give (changing the sign) nx^2y^{-1} for the first Term of the Denominator. Also the second Term will then be ayx^0 , which multiply'd by the Index of y increas'd by n , that is by $n+1$, and divided by y , will give (changing the Sign) $-\overline{n+1a}$ for the second Term of the Denominator, as found above. Now from this general relation of the Fluxions, we may deduce as many particular ones as we please. Thus if we make $m = 0$, and $n = 0$, we shall have $\frac{\dot{y}}{\dot{x}} = \frac{2x}{a}$, or $ay = 2x\dot{x}$, agreeable to our Author's Solution in the place before cited. Or if we make $m = -2$, and $n = -1$, we shall have $\frac{\dot{y}}{\dot{x}} = \frac{2ayx^{-1}}{x^2y^{-1}} = \frac{2ay^2}{x^3}$. Or if we make $m = 0$, and $n = -1$, we shall have $\frac{\dot{y}}{\dot{x}} = \frac{2x}{x^2y^{-1}} = \frac{2y}{x}$. Or if we make $n = 0$, and $m = -2$, we shall have $\frac{\dot{y}}{\dot{x}} = \frac{2ax^{-1}}{a} = \frac{2y}{x}$, as before. All which, and innumerable other cases, may be easily proved by a substitution of equivalents. Or we may prove it generally

rally thus. Because by the given Equation it is $y = x^2 a^{-1}$, in the value of the ratio $\frac{\dot{y}}{\dot{x}} = \frac{mayx^{-1} - m + 2x}{nx^2y^{-1} - n + 1a}$ for y substitute its value, and we shall have $\frac{\dot{y}}{\dot{x}} = \frac{mx - m + 2x}{na - n + 1a} = \frac{2x}{a}$, as above.

3. The Equation of the second Example is $2y^3 + x^2y - 2cyz + 3yz^2 - z^3 = 0$, in which there are three flowing quantities y, x , and z , and therefore there must be three operations, or three Tables must be form'd. First dispose the Terms according to y , thus;

$$2y^3 + 0y^2 + x^2y - z^3y^0 = 0, \text{ and multiply by the Terms of the Pro-}$$

gression $2 \times yy^{-1}, 1 \times yy^{-1}, 0 \times yy^{-1}, -1 \times yy^{-1}$, respectively, (where the Coefficients are form'd by diminishing the Indices of y by the common Number 1,) and the resulting Terms will be $4yy^2 + z^3yy^{-1}$. Secondly, dispose the Terms according to x , thus; $yx^2 + 0x + 2y^3x^0 = 0$,

$$\begin{aligned} & - 2cyz \\ & + 3yz^2 \\ & - z^3 \end{aligned}$$

and multiply by the Terms of the Progression $2 \times \dot{x}x^{-1}, 1 \times \dot{x}x^{-1}, 0 \times \dot{x}x^{-1}$, (where the Coefficients are the same as the Indices of x), and the only resulting Term here is $+ 2y\dot{x}x$. Lastly, dispose the Terms according to z , thus; $-z^3 + 3yz^2 - 2cyz + x^2yz^0 = 0$,

and multiply by the Progression $3 \times \dot{z}z^{-1}, 2 \times \dot{z}z^{-1}, 1 \times \dot{z}z^{-1}, 0 \times \dot{z}z^{-1}$, (where the Coefficients are also the same as the Indices of z), and the Terms will be $-3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z}$. Then collecting all these Terms together, we shall have the Fluxional Equation $4yy^2 + z^3yy^{-1} + 2y\dot{x}x - 3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z} = 0$.

Here we have a notable instance of our Author's dexterity, at finding expedients for abbreviating. For in every one of these Operations such a Progression is chose, as by multiplication will make the greatest destruction of the Terms. By which means he arrives at the shortest Expression, that the nature of the Problem will allow. If we should seek the Fluxions of this Equation by the usual method, which is taught above, that is, if we always assume the Progressions of the Indices, we shall have $6yy^2 + 2\dot{x}xy + x^2\dot{y} - 2cy\dot{z} - 2cy\dot{z} + 3yz^2 + 6y\dot{z}z - 3\dot{z}z^2 = 0$; which has two Terms more than the other form. And if the Progressions of the Indices are increas'd, in each case, by any common general Numbers, we may form the most general Expression for the Fluxional Equation, that the Problem will admit of.

4. On occasion of the last Example, in which are three Fluents and their Fluxions, our Author makes an useful Observation, for the Reduction and compleat Determination of such Equations, tho' it be derived from the Rules of the vulgar Algebra; which matter may be consider'd thus. Every Equation, consisting of two flowing or variable Quantities, is what corresponds to an indetermin'd Problem, admitting of an infinite number of Answers. Therefore one of those quantities being assumed at pleasure, or a particular value being assign'd to it, the other will also be compleatly determined. And in the Fluxional Equation derived from thence, those particular values being substituted, the Ratio of the Fluxions will be given in Numbers, in any particular case. And one of the Fluxions being taken for Unity, or of any determinate value, the value of the other may be exhibited by a Number, which will be a compleat Determination.

But if the given Equation involve three flowing or indeterminate Quantities, two of them must be assumed to determine the third; or, which is the same thing, some other Equation must be either given or assumed, involving some or all the Fluents, in order to a compleat Determination. For then, by means of the two Equations, one of the Fluents may be eliminated, which will bring this to the former case. Also two Fluxional Equations may be derived, involving the three Fluxions, by means of which one of them may be eliminated. And so if the given Equation should involve four Fluents, two other Equations should be either given or assumed, in order to a compleat Determination. This will be sufficiently explain'd by the two following Examples, which will also teach us how complicate Terms, such as compound Fractions and Surds, are to be managed in this Method.

5, 6. Let the given Equation be $y^2 - a^2 - x\sqrt{a^2 - x^2} = 0$, of which we are to take the Fluxions. To the two Fluents y and x we may introduce a third z , if we assume another Equation. Let that be $z = x\sqrt{a^2 - x^2}$, and we shall have the two Equations $y^2 - a^2 - z = 0$, and $a^2x^2 - x^4 - z^2 = 0$. Then by the foregoing Solution their Fluxional Equations (at least in one case) will be $2yy - \dot{z} = 0$, and $a^2\dot{x}x - 2\dot{x}x^3 - \dot{z}z = 0$. These two Fluential Equations, and their Fluxional Equations, may be reduced to one Fluential and one Fluxional Equation, by the usual methods of Reduction: that is, we may eliminate z and $\dot{z}$ by substituting their values $yy - aa$ and $2yy$. Then we shall have $y^2 - a^2 - x\sqrt{a^2 - x^2} = 0$,
 $= 0$,

$= 0$, and $2\dot{y}y = \frac{a^2\dot{x} - 2\dot{x}x^2}{\sqrt{a^2 - x^2}} = 0$. Or by taking away the surds, 'tis $a^2x^2 - x^4 - y^4 + 2a^2y^2 - a^4 = 0$, and then $a^2\dot{x}x - 2\dot{x}x^2 - 2\dot{y}y^3 + 2a^2\dot{y}y = 0$.

7. Or if the given Equation be $x^3 - ay^2 + \frac{b^3}{a+y} - x^2\sqrt{ay+xx} = 0$, to find its corresponding Fluxional Equation; to the two flowing quantities x and y we may introduce two others z and v , and thereby remove the Fraction and the Radical, if we assume the two Equations $\frac{b^3}{a+y} = z$, and $x^2\sqrt{ay+xx} = v$. Then we shall have the three Equations $x^3 - ay^2 + z - v = 0$, $ax + yz - by^3 = 0$, and $ayx^4 + x^6 - v^2 = 0$, which will give the three Fluxional Equations $3\dot{x}x^2 - 2a\dot{y}y + \dot{z} - \dot{v} = 0$, $a\dot{x} + y\dot{z} + y\dot{z} - 3b\dot{y}y^2 = 0$, and $ayx^4 + 4ay\dot{x}x^3 + 6\dot{x}x^5 - 2\dot{v}v = 0$. These by known Methods of the common Algebra may be reduced to one Fluential and one Fluxional Equation, involving x and y , and their Fluxions, as is required.

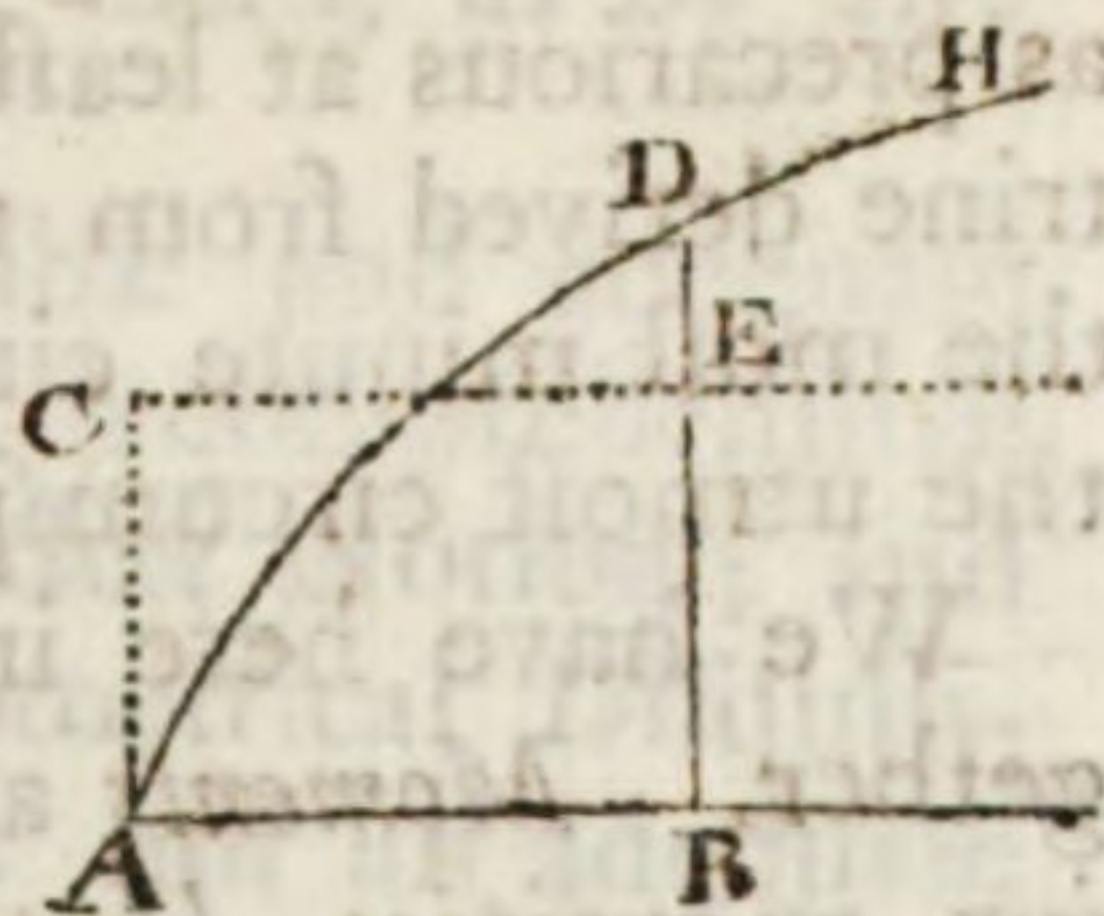
8. And by the same Method we may take the Fluxions of Binomial or other Radicals, of any kind, any how involved or complicated with one another. As for instance, if we were to find the Fluxion of $\sqrt{ax} + \sqrt{aa - xx}$, put it equal to y , or make $ax + \sqrt{aa - xx} = yy$. Also make $\sqrt{aa - xx} = z$. Then we shall have the two Fluential Equations $ax + z - y^2 = 0$, and $a^2 - x^2 - z^2 = 0$, from whence we shall have the two Fluxional Equations $a\dot{x} + \dot{z} - 2\dot{y}y = 0$, and $-2\dot{x}x - 2\dot{z}z = 0$, or $\dot{x}x + \dot{z}z = 0$. This last Equation, if for z and $\dot{z}$ we substitute their values $yy - ax$ and $2\dot{y}y - a\dot{x}$, will become $x\dot{x} + 2\dot{y}y^3 - 2ax\dot{y}y - a\dot{x}y^2 + a^2\dot{x}x = 0$; whence $\dot{y} = \frac{a\dot{x}y^2 - a^2\dot{x}x - \dot{x}x}{2y^3 - 2axy}$. And here if for y we substitute its value $\sqrt{ax} + \sqrt{aa - xx}$, we shall have the Fluxion required $\dot{y} = \frac{a\dot{x}\sqrt{aa - xx} - \dot{x}x}{2\sqrt{aa - xx} \times \sqrt{ax} + \sqrt{aa - xx}}$. And many other Examples of a like kind will be found in the sequel of this Work.

9, 10, 11, 12. In Examp. 5. the proposed Equation is $zz + axz - y^4 = 0$, in which there are three variable quantities x , y , and z , and therefore the relation of the Fluxions will be $2\dot{z}z + ax\dot{z} + a\dot{x}z - 4\dot{y}y^3 = 0$. But as there wants another Fluential Equation, and thence another Fluxional Equation, to make a compleat determination; if only another Fluxional Equation were given or assumed, we should have the required relation of the Fluxions $\dot{x}$ and $\dot{y}$.

Suppose

Suppose this Fluxional Equation were $\dot{z} = \dot{x}\sqrt{ax - xx}$; then by substitution we should have the Equation $2z + ax \times \dot{x}\sqrt{ax - xx} + axz - 4yy^2 = 0$, or the Analogy $\dot{x} : y :: 4y^3 : 2z + ax \times \sqrt{ax - xx} + az$, which can be reduced no farther, (or z cannot be eliminated,) till we have the Fluential Equation, from which the Fluxional Equation $\dot{z} = \dot{x}\sqrt{ax - xx}$ is suppos'd to be derived. And thus we may have the relation of the Fluxions, even in such cases as we have not, or perhaps cannot have, the relation of the Fluents.

But tho' this Reduction may not perhaps be conveniently perform'd Analytically, or by Calculation, yet it may possibly be perform'd Geometrically, as it were, and by the Quadrature of Curves; as we may learn from our Author's preparatory Proposition, and from the following general Considerations. Let the right Line AC, perpendicular to the right Line AB, be conceived to move always parallel to itself, so as that its extremity A may describe the line AB. Let the point C be fixt, or always at the same distance from A, and let another point move from A towards C, with a velocity any how accelerated or retarded. The parallel motion of the line AC does not at all affect the progressive motion of the point moving from A towards C, but from a combination of these two independent motions, it will describe the Curve ADH; while at the same time the fixt point C will describe the right line CE, parallel to AB. Let the line AC be conceived to move thus, till it comes into the place BE, or BD. Then the line AC is constant, and remains the same, while the indefinite or flowing line becomes BD. Also the Areas described at the same time, ACEB and ADB, are likewise flowing quantities, and their velocities of description, or their Fluxions, must necessarily be as their respective describing lines, or Ordinates, BE and BD. Let AC or BE be Linear Unity, or a constant known right line, to which all the other lines are to be compared or refer'd; just as in Numbers, all other Numbers are tacitely refer'd to 1, or to Numeral Unity, as being the simplest of all Numbers. And let the Area ADB be suppos'd to be apply'd to BE, or Linear Unity, by which it will be reduced from the order of Surfaces to that of Lines; and let the resulting line be call'd z . That is, make the Area ADB $= z \times BE$; and if AB be call'd x , then is the Area ACEB $= x \times BE$. Therefore the



Fluxions of these Areas will be $\dot{z} \times BE$ and $\dot{x} \times BE$, which are as $\dot{z}$ and $\dot{x}$. But the Fluxions of the Areas were found before to be as BD to BE . So that it is $\dot{z} : \dot{x} :: BD : BE = 1$, or $\dot{z} = \dot{x} \times BD$. Consequently in any Curve, the Fluxion of the Area will be as the Ordinate of the Curve, drawn into the Fluxion of the Absciss.

Now to apply this to the present case. In the Fluxional Equation before assumed $\dot{z} = \dot{x} \sqrt{ax - xx}$, if x represents the Absciss of a Curve, and $\sqrt{ax - xx}$ be the Ordinate; then will this Curve be a Circle, and z will represent the corresponding Area. So that we see from hence, whether the Area of a Circle can be exhibited or no, or, in general Terms, tho' in the Equation proposed there should be quantities involved, which cannot be determined or express'd by any Geometrical Method, such as the Areas or Lengths of Curve-lines; yet the relation of their Fluxions may nevertheless be found.

13. We now come to the Author's Demonstration of his Solution, or to the proof of the Principles of the Method of Fluxions, here laid down, which certainly deserves to engage our most serious attention. And more especially, because these Principles have been lately drawn into debate, without being well consider'd or understood; possibly because this Treatise of our Author's, expressly wrote on the subject, had not yet seen the light. As these Principles therefore have been treated as precarious at least, if not wholly insufficient to support the Doctrine derived from them; I shall endeavour to examine into every the most minute circumstance of this Demonstration, and that with the utmost circumspection and impartiality.

We have here in the first place a Definition and a Theorem together. *Moments* are defined to be *the indefinitely small parts of flowing quantities, by the accession of which, in indefinitely small portions of time, those quantities are continually increased.* The word *Moment* (*momentum, movimentum, à moveo,*) by analogy seems to have been borrow'd from Time. For as Time is conceived to be in continual flux, or motion, and as a greater and a greater Time is generated by the accession of more and more Moments, which are conceived as the smallest particles of Time: So all other flowing Quantities may be understood as perpetually increasing, by the accession of their smallest particles, which therefore may not improperly be call'd their Moments. But what are here call'd their *smallest particles*, are not to be understood as if they were Atoms, or of any definite and determinate magnitude, as in the Method of Indivisibles; but to be indefinitely small, or continually decreasing, till they are less than

than any assignable quantities, and yet may then retain all possible varieties of proportion to one another. That these Moments are not chimerical, visionary, or merely imaginary things, but have an existence *sui generis*, at least Mathematically and in the Understanding, is a necessary consequence from the infinite Divisibility of Quantity, which I think hardly any body now contests *. For all continued quantity whatever, tho' not indeed actually, yet mentally may be conceived to be divided *in infinitum*. Perhaps this may be best illustrated by a comparative gradation or progress of Magnitudes. Every finite and limited Quantity may be conceived as divided into any finite number of smaller parts. This Division may proceed, and those parts may be conceived to be farther divided in very little, but still finite parts, or *particles*, which yet are not Moments. But when these particles are farther conceived to be divided, not actually but mentally, so far as to become of a magnitude less than any assignable, (and what can stop the progress of the Mind?) then are they properly the *Moments* which are to be understood here. As this gradation of diminution certainly includes no absurdity or contradiction, the Mind has the privilege of forming a Conception of these Moments, a possible Notion at least, though perhaps not an adequate one; and then Mathematicians have a right of applying them to use, and of making such Inferences from them, as by any strict way of reasoning may be derived.

It is objected, that we cannot form an intelligible and adequate Notion of these Moments, because so obscure and incomprehensible an Idea, as that of Infinity is, must needs enter that Notion; and therefore they ought to be excluded from all Geometrical Disquisitions. It may indeed be allowed, that we have not an adequate Notion of them on that account, such as exhausts the whole nature of the thing, neither is it at all necessary; for a partial Notion, which is that of their Divisibility *sine fine*, without any regard to their magnitude, is sufficient in the present case. There are many other Speculations in the Mathematicks, in which a Notion of Infinity is a necessary ingredient, which however are admitted by all Geometricians, as useful and demonstrable Truths. The Doctrine of commensurable and incommensurable magnitudes includes a Notion of Infinity, and yet is received as a very demonstrable Doctrine. We have a perfect Idea of a Square and its Diagonal, and yet we

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know

* Perhaps the ingenious Author of the Discourse call'd *The Analyst* must be excepted, who is pleas'd to ask, in his fifth Query, *Whether it be not unnecessary, as well as absurd, to suppose that finite Extension is infinitely divisible?* See also Query 19, 20, 21, &c.

know they will admit of no finite common measure, or that their proportion cannot be exhibited in rational Numbers, tho' ever so small, but may by a series of decimal or other parts continued *ad infinitum*. In common Arithmetick we know, that the vulgar Fraction $\frac{2}{3}$, and the decimal Fraction 0,666666, &c. continued *ad infinitum*, are one and the same thing; and therefore if we have a scientifick notion of the one, we have likewise of the other. When I describe a right line with my Pen, suppose of an Inch long, I describe first one half of the line, then one half of the remainder, then one half of the next remainder, and so on. That is, I actually run over all those infinite divisions and subdivisions, before I have completed the Line, tho' I do not attend to them, or cannot distinguish them. And by this I am indubitably certain, that this Series of Fractions $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, &c. continued *ad infinitum*, is precisely equal to Unity. *Euclid* has demonstrated in his Elements, that the Circular Angle of Contact is less than any assignable right-lined Angle, or, which is the same thing, is an infinitely little Angle in comparison with any finite Angle: And our Author shews us still greater Mysteries, about the infinite gradations of Angles of Contact. In Geometry we know, that Curves may continually approach towards their Asymptotes, and yet will not actually meet with them, till both are continued to an infinite distance. We know likewise, that many of their included Areas, or Solids, will be but of a finite and determinable magnitude, even tho' their lengths should be actually continued *ad infinitum*. We know that some Spirals make infinite Circumvolutions about a Pole, or Center, and yet the whole Line, thus infinitely involved, is but of a finite, determinable, and assignable length. The Methods of computing Logarithms suppose, that between any two given Numbers, an infinite number of mean Proportionals may be interposed; and without some Notion of Infinity their nature and properties are hardly intelligible or discoverable. And in general, many of the most sublime and useful parts of knowledge must be banish'd out of the Mathematicks, if we are so scrupulous as to admit of no Speculations, in which a Notion of Infinity will be necessarily included. We may therefore as safely admit of Moments, and the Principles upon which the Method of Fluxions is here built, as any of the fore-mention'd Speculations.

The nature and notion of Moments being thus establish'd, we may pass on to the afore-mention'd Theorem, which is this.

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The (contemporary) Moments of flowing quantities are as the Velocities of flowing or increasing; that is, as their Fluxions. Now if this be proved of Lines, it will equally obtain in all flowing quantities whatever, which may always be adequately represented and expounded by Lines. But in equable Motions, the Times being given, the Spaces described will be as the Velocities of Description, as is known in Mechanicks. And if this be true of any finite Spaces whatever, or of all Spaces in general, it must also obtain in infinitely little Spaces, which we call Moments. And even in Motions continually accelerated or retarded, the Motions in infinitely little Spaces, or Moments, must degenerate into equability. So that the Velocities of increase or decrease, or the Fluxions, will be always as the contemporary Moments. Therefore the Ratio of the Fluxions of Quantities, and the Ratio of their contemporary Moments, will always be the same, and may be used promiscuously for each other.

14. The next thing to be settled is a convenient Notation for these Moments, by which they may be distinguish'd, represented, compared, and readily suggested to the Imagination. It has been appointed already, that when $x, y, z, v, \&c.$ stand for variable or flowing quantities, then their Velocities of increase, or their Fluxions, shall be represented by $\dot{x}, \dot{y}, \dot{z}, \dot{v}, \&c.$ which therefore will be proportional to the contemporary Moments. But as these are only Velocities, or magnitudes of another Species, they cannot be the Moments themselves, which we conceive as indefinitely little Spaces, or other analogous quantities. We may therefore here aptly introduce the Symbol o , not to stand for absolute nothing, as in Arithmetick, but a vanishing Space or Quantity, which was just now finite, but by continually decreasing, in order presently to terminate in mere nothing, is now become less than any assignable Quantity. And we have certainly a right so to do. For if the notion is intelligible, and implies no contradiction as was argued before, it may surely be insinuated by a Character appropriate to it. This is not assigning the quantity, which would be contrary to the *hypothesis*, but is only appointing a mark to represent it. Then multiplying the Fluxions by the vanishing quantity o , we shall have the several quantities $\dot{x}o, \dot{y}o, \dot{z}o, \dot{v}o, \&c.$ which are vanishing likewise, and proportional to the Fluxions respectively. These therefore may now represent the contemporary Moments of $x, y, z, v, \&c.$ And in general, whatever other flowing quantities, as well as Lines and Spaces,

Spaces, are represented by $x, y, z, v, \&c.$ as o may stand for a vanishing quantity of the same kind, and as $\dot{x}, \dot{y}, \dot{z}, \dot{v}, \&c.$ may stand for their Velocities of increase or decrease, (or, if you please, for Numbers proportional to those Velocities,) then may $\dot{x}o, \dot{y}o, \dot{z}o, \dot{v}o, \&c.$ always denote their respective synchronal Moments, or momentary accessions, and may be admitted into Computations accordingly. And this we come now to apply.

15. We must now have recourse to a very notable, useful, and extensive property, belonging to all Equations that involve flowing Quantities. Which property is, that in the progress of flowing, the Fluents will continually acquire new values, by the accession of contemporary parts of those Fluents, and yet the Equation will be equally true in all these cases. This is a necessary result from the Nature and Definition of variable Quantities. Consequently these Fluents may be any how increased or diminish'd by their contemporary Increments or Decrements; which Fluents, so increased or diminished, may be substituted for the others in the Equation. As if an Equation should involve the Fluents x and y , together with any given quantities, and X and Y are supposed to be any of their contemporary Augments respectively. Then in the given Equation we may substitute $x + X$ for x , and $y + Y$ for y , and yet the Equation will be good, or the equality of the Terms will be preserved. So if X and Y were contemporary Decrements, instead of x and y we might substitute $x - X$ and $y - Y$ respectively. And as this must hold good of all contemporary Increments or Decrements whatever, whether finitely great or infinitely little, it will be true likewise of contemporary Moments. That is, instead of x and y in any Equation, we may substitute $x + \dot{x}o$ and $y + \dot{y}o$, and yet we shall still have a good Equation. The tendency of this will appear from what immediately follows.

16. The Author's single Example is a kind of Induction, and the proof of this may serve for all cases. Let the Equation $x^3 - ax^2 + axy - y^3 = 0$ be given as before, including the variable quantities x and y ; instead of which we may substitute these quantities increas'd by their contemporary Moments, or $x + \dot{x}o$ and $y + \dot{y}o$ respectively. Then we shall have the Equation $\overline{x + \dot{x}o}^3 - a \times \overline{x + \dot{x}o}^2 + a \times \overline{x + \dot{x}o} \times \overline{y + \dot{y}o} - \overline{y + \dot{y}o}^3 = 0$. These Terms being expanded, and reduced to three orders or columns, according as the vanishing quantity o is of none, one, or of more dimensions, will stand as in the Margin.

17, 18. Here the Terms of the first order, or column, remove or destroy one another, as being absolutely equal to nothing by the given Equation. They being therefore expunged, the remaining Terms may all be divided by the common Multiplier o , whatever it is. This being done, all the Terms of the third order will still be affected by o , of one or more dimensions, and may therefore be expunged, as infinitely less than the others. Lastly, there will only remain those of the second order or column, that is $3xx^2 - 2axx + axy + ayx - 3yy^2 = 0$, which will be the Fluxional Equation required. Q. E. D.

$$\left. \begin{array}{r} x^3 + 3xx^2 + 3x^2x + x^3 \\ - ax^2 - 2axx - ax^2 \\ + axy + axy + axy^2 \\ + ayx \\ - y^3 - 3yy^2 - 3y^2y \\ - y^3 \end{array} \right\} = 0.$$

The same Conclusions may be thus derived, in something a different manner. Let X and Y be any synchroal Augments of the variable quantities x and y , as before, the relation of which quantities is exhibited by any Equation. Then may $x + X$ and $y + Y$ be substituted for x and y in that Equation. Suppose for instance that $x^3 - ax^2 + axy - y^3 = 0$; then by substitution we shall have $(x + X)^3 - a(x + X)^2 + a(x + X)(y + Y) - (y + Y)^3 = 0$; or in *terminis expansis* $x^3 + 3x^2X + 3xX^2 + X^3 - ax^2 - 2axX - aX^2 + axy + axY + aXy + aXY - y^3 - 3y^2Y - 3yY^2 - Y^3 = 0$. But the Terms $x^3 - ax^2 + axy - y^3 = 0$ will vanish out of the Equation, and leave $3x^2X + 3xX^2 + X^3 - 2axX - aX^2 + axY + aXy + aXY - 3y^2Y - 3yY^2 - Y^3 = 0$, for the relation of the contemporary Augments, let their magnitude be what it will. Or resolving this Equation into an Analogy, the ratio of these Augments may be this, $\frac{Y}{X} = \frac{3x^2 + 3xX + X^2 - 2ax - aX + ay}{-ax - aX + 3y^2 + 3yY + Y^2}$.

Now to find the *ultimate ratio* of these Augments, or their ratio when they become Moments, suppose X and Y to diminish till they become vanishing quantities, and then they may be expunged out of this value of the ratio. Or in those circumstances it will be $\frac{Y}{X} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}$, which is now the ratio of the Moments. And this is the same ratio as that of the Fluxions, or it will be $\frac{\dot{y}}{\dot{x}} = \frac{3x^2 - 2ax + ay}{3y^2 - ax}$, or $3y^2\dot{y} - ax\dot{y} = 3x^2\dot{x} - 2ax\dot{x} + ay\dot{x}$, as was found before.

In this way of arguing there is no assumption made, but what is justifiable by the received Methods both of the ancient and modern Geometricians. We only descend from a general Proposition, which is undeniable, to a particular case which is certainly included in it. That

That is, having the relation of the variable Quantities, we thence directly deduce the relation or ratio of their contemporary Augments; and having this, we directly deduce the relation or ratio of those contemporary Augments when they are nascent or evanescent, just beginning or just ceasing to be; in a word, when they are Moments, or vanishing Quantities. To evade this reasoning, it ought to be proved, that no Quantities can be conceived less than assignable Quantities; that the Mind has not the privilege of conceiving Quantity as perpetually diminishing *sine fine*; that the Conception of a vanishing Quantity, a Moment, an Infinitesimal, &c. includes a contradiction: In short, that Quantity is not (even mentally) divisible *ad infinitum*; for to that the Controversy must be reduced at last. But I believe it will be a very difficult matter to extort this Principle from the Mathematicians of our days, who have been so long in quiet possession of it, who are indubitably convinced of the evidence and certainty of it, who continually and successfully apply it, and who are ready to acknowledge the extreme fertility and usefulness of it, upon so many important occasions.

19. Nothing remains, I think, but to account for these two circumstances, belonging to the Method of Fluxions, which our Author briefly mentions here. First that the given Equation, whose Fluxional Equation is to be found, may involve any number of flowing quantities. This has been sufficiently proved already, and we have seen several Examples of it. Secondly, that in taking Fluxions we need not always confine ourselves to the progression of the Indices, but may assume infinite other Arithmetical Progressions, as conveniency may require. This will deserve a little farther illustration, tho' it is no other than what must necessarily result from the different forms, which any given Equation may assume, in an infinite variety. Thus the Equation $x^3 - ax^2 + axy - y^3 = 0$, being multiply'd by the general quantity $x^m y^n$, will become $x^{m+3} y^n - ax^{m+2} y^n + ax^{m+1} y^{n+1} - x^m y^{n+3} = 0$, which is virtually the same Equation as it was before, tho' it may assume infinite forms, according as we please to interpret m and n . And if we take the Fluxions of this Equation, in the usual way, we shall have $m + 3xx^{m+2}y^n + nx^{m+3}yy^{n-1} - m + 2axx^{m+1}y^n - nax^{m+2}yy^{n-1} + m + 1axx^m y^{n+1} + n + 1ax^{m+1}yy^n - mxx^{m-1}y^{n+3} - n + 3x^m yy^{n+2} = 0$. Now if we divide this again by $x^m y^n$, we shall have $m + 3xx^2 + nx^3yy^{-1} - m + 2axx - nax^2yy^{-1} + m + 1axy + n + 1axy - mxx^{-1}y^3 - n + 3yy^2 = 0$, which is the same general Equation as was derived before. And the like may be understood of all other Examples.

SECT. II. *Concerning Fluxions of superior orders, and the method of deriving their Equations.*

IN this Treatise our Author considers only first Fluxions, and has not thought fit to extend his Method to superior orders, as not directly falling within his present purpose. For tho' he here pursues Speculations which require the use of second Fluxions, or higher orders, yet he has very artfully contrived to reduce them to first Fluxions, and to avoid the necessity of introducing Fluxions of superior orders. In his other excellent Works of this kind, which have been publish'd by himself, he makes express mention of them, he discovers their nature and properties, and gives Rules for deriving their Equations. Therefore that this Work may be the more serviceable to Learners, and may fulfil the design of being an Institution, I shall here make some inquiry into the nature of superior Fluxions, and give some Rules for finding their Equations. And afterwards, in its proper place, I shall endeavour to shew something of their application and use.

Now as the Fluxions of quantities which have been hitherto consider'd, or their comparative Velocities of increase and decrease, are themselves, and of their own nature, variable and flowing quantities also, and as such are themselves capable of perpetual increase and decrease, or of perpetual acceleration and retardation; they may be treated as other flowing quantities, and the relation of their Fluxions may be inquired and discover'd. In order to which we will adopt our Author's Notation already publish'd, in which we are to conceive, that as $x, y, z, \&c.$ have their Fluxions $\dot{x}, \dot{y}, \dot{z}, \&c.$ so these likewise have their Fluxions $\ddot{x}, \ddot{y}, \ddot{z}, \&c.$ which are the second Fluxions of $x, y, z, \&c.$ And these again, being still variable quantities, have their Fluxions denoted by $\dddot{x}, \dddot{y}, \dddot{z}, \&c.$ which are the third Fluxions of $x, y, z, \&c.$ And these again, being still flowing quantities, have their Fluxions $\ddddot{x}, \ddddot{y}, \ddddot{z}, \&c.$ which are the fourth Fluxions of $x, y, z, \&c.$ And so we may proceed to superior orders, as far as there shall be occasion. Then, when any Equation is proposed, consisting of variable quantities, as the relation of its Fluxions may be found by what has been taught before; so by repeating only the same operation, and considering the Fluxions as flowing Quantities, the

relation of the second Fluxions may be found. And the like for all higher orders of Fluxions.

Thus if we have the Equation $y^2 - ax = 0$, in which are the two Fluents y and x , we shall have the first Fluxional Equation $2\dot{y}\dot{y} - a\dot{x} = 0$. And here, as we have the three Fluents y , $\dot{y}$, and $\dot{x}$; if we take the Fluxions again, we shall have the second Fluxional Equation $2\ddot{y}\dot{y} + 2\dot{y}^2 - a\ddot{x} = 0$. And here, as there are four Fluents y , $\dot{y}$, $\ddot{y}$, and $\ddot{x}$, if we take the Fluxions again, we shall have the third Fluxional Equation $2\ddot{y}\dot{y} + 2\ddot{y}\dot{y} + 4\dot{y}\ddot{y} - a\ddot{x} = 0$, or $2\ddot{y}\dot{y} + 6\dot{y}\ddot{y} - a\ddot{x} = 0$. And here, as there are five Fluents y , $\dot{y}$, $\ddot{y}$, $\ddot{y}$, and $\ddot{x}$, if we take the Fluxions again, we shall have the fourth Fluxional Equation $2\ddot{y}\dot{y} + 2\ddot{y}\dot{y} + 6\dot{y}\ddot{y} + 6\dot{y}^2 - a\ddot{x} = 0$, or $2\ddot{y}\dot{y} + 8\dot{y}\ddot{y} + 6\dot{y}^2 - a\ddot{x} = 0$. And here, as there are six Fluents y , $\dot{y}$, $\ddot{y}$, $\ddot{y}$, $\ddot{y}$, and $\ddot{x}$, if we take the Fluxions again, we shall have $2\ddot{y}\dot{y} + 2\ddot{y}\dot{y} + 8\dot{y}\ddot{y} + 8\dot{y}^2 + 12\ddot{y}\dot{y} - a\ddot{x} = 0$, or $2\ddot{y}\dot{y} + 10\dot{y}\ddot{y} + 20\ddot{y}\dot{y} - a\ddot{x} = 0$, for the fifth Fluxional Equation. And so on to the sixth, seventh, &c.

Now the Demonstration of this will proceed much after the manner as our Author's Demonstration of first Fluxions, and is indeed virtually included in it. For in the given Equation $y^2 - ax = 0$, if we suppose y and x to become at the same time $y + \dot{y}o$ and $x + \dot{x}o$, (that is, if we suppose $\dot{y}o$ and $\dot{x}o$ to denote the synchronal Moments of the Fluents y and x ;) then by substitution we shall have $y + \dot{y}o - a(x + \dot{x}o) = 0$, or *in terminis expansis*, $y^2 + 2y\dot{y}o + \dot{y}^2o^2 - ax - a\dot{x}o = 0$. Where expunging $y^2 - ax = 0$, and $\dot{y}^2o^2$, and dividing the rest by o , it will be $2y\dot{y} - a\dot{x} = 0$ for the first fluxional Equation. Now in this Equation, if we suppose the synchronal Moments of the Fluents y , $\dot{y}$, and $\dot{x}$, to be $\dot{y}o$, $\ddot{y}o$, and $\ddot{x}o$ respectively; for those Fluents we may substitute $y + \dot{y}o$, $\dot{y} + \ddot{y}o$, and $\dot{x} + \ddot{x}o$ in the last Equation, and it will become $2y + 2\dot{y}o \times \dot{y} + \dot{y}o - a\dot{x} + \ddot{x}o = 0$, or expanding, $2y\dot{y} + 2\dot{y}\dot{y}o + 2y\ddot{y}o + 2\dot{y}\ddot{y}oo - a\dot{x} - a\ddot{x}o = 0$. Here because $2y\dot{y} - a\dot{x} = 0$ by the given Equation, and because $2\dot{y}\ddot{y}oo$ vanishes; divide the rest by o , and we shall have $2\dot{y}^2 + 2y\ddot{y} - a\ddot{x} = 0$ for the second fluxional Equation. Again in this Equation, if we suppose the Synchronal Moments of the Fluents y , $\dot{y}$, $\ddot{y}$, and $\ddot{x}$, to be $\dot{y}o$, $\ddot{y}o$, $\ddot{y}o$, and $\ddot{x}o$ respectively; for those Fluents we

we may substitute $y + \dot{y}o$, $\dot{y} + \ddot{y}o$, $\ddot{y} + \dddot{y}o$, and $\ddot{x} + \dddot{x}o$ in the last Equation, and it will become $2 \times \dot{y} + \ddot{y}o \mid^2 + 2y + 2\dot{y}o \times \ddot{y} + \ddot{y}o - a \times \ddot{x} + \ddot{x}o = 0$, or expanding and collecting, $2\dot{y}^2 + 6\dot{y}\ddot{y}o + 2\ddot{y}^2o^2 + 2y\ddot{y} + 2\dot{y}\ddot{y}o + 2\ddot{y}\ddot{y}o^2 - a\ddot{x} - a\ddot{x}o = 0$. But here because $2\dot{y}^2 + 2y\ddot{y} - a\ddot{x} = 0$ by the last Equation; dividing the rest by o , and expunging all the Terms in which o will still be found, we shall have $6\dot{y}\ddot{y} + 2\ddot{y}^2 - a\ddot{x} = 0$ for the third fluxional Equation. And in like manner for all other orders of Fluxions, and for all other Examples. Q. E. D.

To illustrate the method of finding superior Fluxions by another Example, let us take our Author's Equation $x^3 - ax^2 + axy - y^3 = 0$, in which he has found the simplest relation of the Fluxions to be $3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + ax\dot{y} - 3\dot{y}y^2 = 0$. Here we have the flowing quantities $x, y, \dot{x}, \dot{y}$; and by the same Rules the Fluxion of this Equation, when contracted, will be $3\ddot{x}x^2 + 6\dot{x}^2x - 2a\ddot{x}x - 2a\dot{x}^2 + a\ddot{x}y + 2a\dot{x}\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$. And in this Equation we have the flowing quantities $x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}$, so that taking the Fluxions again by the same Rules, we shall have the Equation, when contracted, $3\ddot{x}x^2 + 18\dot{x}\ddot{x}x + 6\dot{x}^3 - 2a\ddot{x}x - 6a\dot{x}\ddot{x} + a\ddot{x}y + 3a\dot{x}\ddot{y} + 3a\ddot{x}\dot{y} + ax\ddot{\ddot{y}} - 3\ddot{y}y^2 - 18\dot{y}\ddot{y}y - 6\dot{y}^3 = 0$. And as in this Equation there are found the flowing quantities $x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \ddot{\ddot{x}}, \ddot{\ddot{y}}$, we might proceed in like manner to find the relations of the fourth Fluxions belonging to this Equation, and all the following orders of Fluxions.

And here it may not be amiss to observe, that as the proposed Equation expresses the constant relation of the variable quantities x and y ; and as the first fluxional Equation expresses the constant relation of the variable (but finite and assignable) quantities $\dot{x}$ and $\dot{y}$, which denote the comparative Velocity of increase or decrease of x and y in the proposed Equation: So the second fluxional Equation will express the constant relation of the variable (but finite and assignable) quantities $\ddot{x}$ and $\ddot{y}$, which denote the comparative Velocity of the increase or decrease of $\dot{x}$ and $\dot{y}$ in the foregoing Equation. And in the third fluxional Equation we have the constant relation of the variable (but finite and assignable) quantities $\ddot{\ddot{x}}$ and $\ddot{\ddot{y}}$, which will denote the

comparative Velocity of the increase or decrease of $\ddot{x}$ and $\ddot{y}$ in the foregoing Equation. And so on for ever. Here the Velocity of a Velocity, however uncouth it may sound, will be no absurd Idea when rightly conceived, but on the contrary will be a very rational and intelligible Notion. If there be such a thing as Motion any how continually accelerated, that continual Acceleration will be the Velocity of a Velocity; and as that variation may be continually varied, that is, accelerated or retarded, there will be in nature, or at least in the Understanding, the Velocity of a Velocity of a Velocity. Or in other words, the Notion of second, third, and higher Fluxions, must be admitted as sound and genuine. But to proceed:

We may much abbreviate the Equations now derived, by the known Laws of Analyticks. From the given Equation $x^3 - ax^2 + axy - y^3 = 0$ there is found a new Equation, wherein, because of two new Symbols $\dot{x}$ and $\dot{y}$ introduced, we are at liberty to assume another Equation, besides this now found, in order to a just Determination. For simplicity-sake we may make $\dot{x}$ Unity, or any other constant quantity; that is, we may suppose x to flow equably, and therefore its Velocity is uniform. Make therefore $\dot{x} = 1$, and the first fluxional Equation will become $3x^2 - 2ax + ay + ax\dot{y} - 3\dot{y}y^2 = 0$. So in the Equation $3\ddot{x}x^2 + 6\dot{x}^2x - 2a\ddot{x}x - 2a\dot{x}^2 + ax\ddot{y} + 2a\dot{x}\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$ there are four new Symbols introduced, $\ddot{x}$, $\dot{y}$, $\ddot{x}$, and $\ddot{y}$, and therefore we may assume two other congruous Equations, which together with the two now found, will amount to a compleat Determination. Thus if for the sake of simplicity we make one to be $\ddot{x} = 1$, the other will necessarily be $\ddot{x} = 0$; and these being substituted, will reduce the second fluxional Equation to this, $6x - 2a + 2a\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$. And thus in the next Equation, wherein there are six new Symbols $\ddot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$, $\ddot{x}$, $\dot{y}$; besides the three Equations now found, we may take $\ddot{x} = 1$, and thence $\ddot{x} = 0$, $\ddot{x} = 0$, which will reduce it to $6 + 3a\ddot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 18\dot{y}\ddot{y}y - 6\dot{y}^3 = 0$. And the like of Equations of succeeding orders.

But all these Reductions and Abbreviations will be best made as the Equations are derived. Thus the proposed Equation being $x^3 - ax^2 + axy - y^3 = 0$, taking the Fluxions, and at the same time making $\dot{x} = 1$, (and consequently $\ddot{x}$, $\ddot{x}$, &c. $= 0$,) we shall have $3x^2 - 2ax + ay + ax\dot{y} - 3\dot{y}y^2 = 0$. And taking the Fluxions again,

again, it will be $6x - 2a + 2a\dot{y} + ax\ddot{y} - 3\ddot{y}y^2 - 6\dot{y}^2y = 0$.

And taking the Fluxions again, it will be $6 + 3a\ddot{y} + ax\dddot{y} - 3\ddot{y}\dot{y}^2 - 18\ddot{y}y\dot{y} - 6\dot{y}^3 = 0$. And taking the Fluxions again, it will be

$4a\ddot{y} + ax\dddot{y} - 3\ddot{y}\dot{y}^2 - 24\ddot{y}y\dot{y} - 18\dot{y}^2y - 36\ddot{y}\dot{y}^2 = 0$. And so on, as far as there is occasion.

But now for the clearer apprehension of these several orders of Fluxions, I shall endeavour to illustrate them by a Geometrical Figure, adapted to a simple and a particular case. Let us assume the Equation $y^2 = ax$, or $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$, which will therefore belong to the Parabola ABC, whose Parameter is $AP = a$, Absciss $AD = x$, and Ordinate $BD = y$; where AP is a Tangent at the Vertex A. Then taking the Fluxions, we shall have $\dot{y} = \frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}$. And supposing the Parabola to be described by the equable motion of the Ordinate upon the Absciss, that equable Velocity may be expounded by the given Line or Parameter a , that is, we may put $\dot{x} = a$. Then

it will be $\dot{y} = (\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{3}{2}}}{2x^{\frac{1}{2}}} = \frac{a^{\frac{3}{2}}x^{\frac{1}{2}}}{2x} =) \frac{ay}{2x}$, which will give us

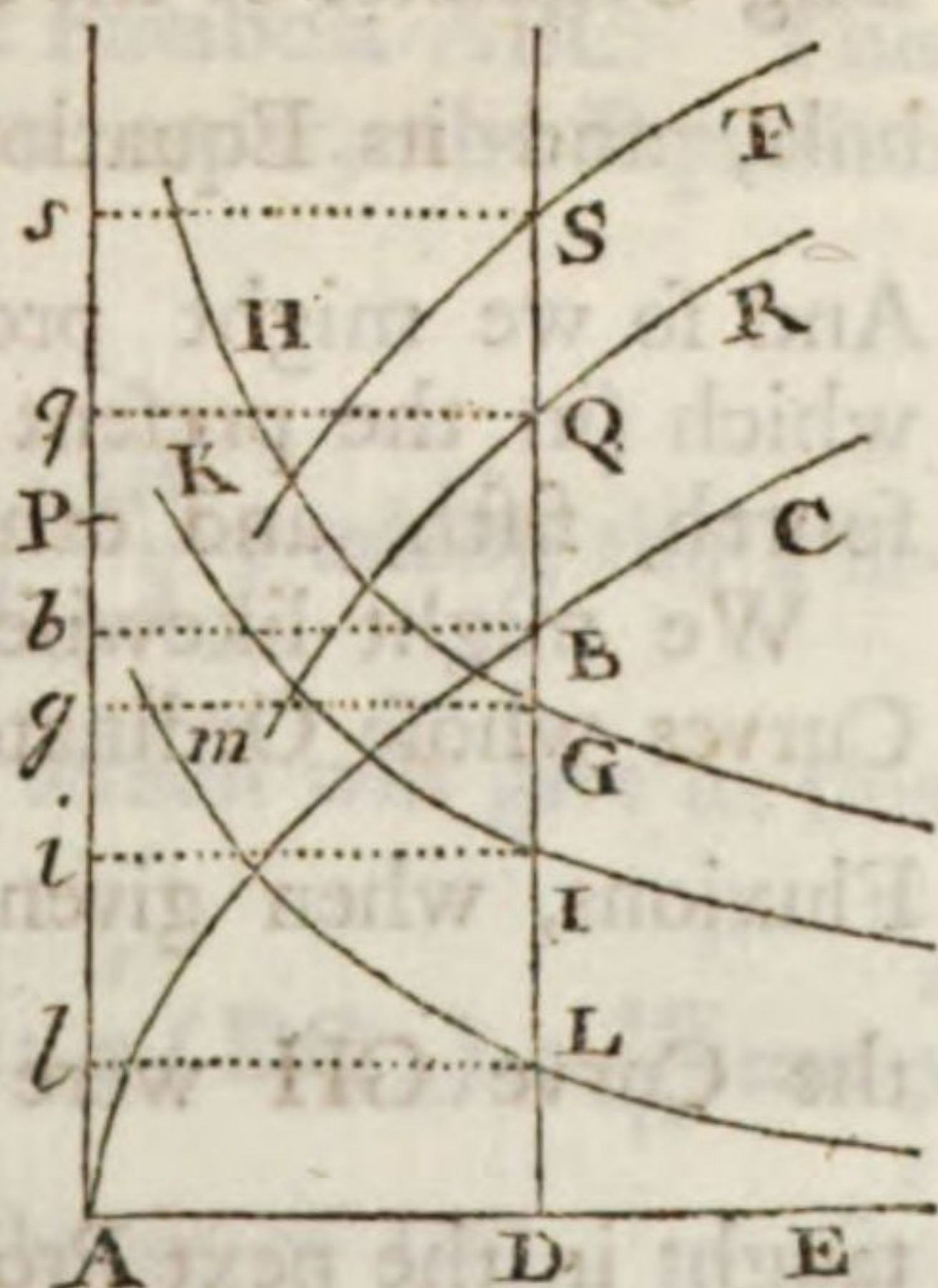
this Construction. Make x (AD) : y (BD) :: $\frac{1}{2}a$ ($\frac{1}{2}AP$) : $DG = \frac{ay}{2x} = \dot{y}$, and the Line DG will therefore

represent the Fluxion of y or BD. And if this be done every where upon AE, (or if the Ordinate DG be suppos'd to move upon AE with a parallel motion,) a Curve GH will be constructed or described, whose Ordinates will every where expound the Fluxions of the corresponding Ordinates of the Parabola ABC. This Curve will be one of the Hyperbola's between the Asymptotes

AE and AP; for its Equation is $\dot{y} = \frac{a^{\frac{3}{2}}}{2x^{\frac{1}{2}}}$,

or $\dot{y}\dot{y} = \frac{a^3}{4x}$.

Again, from the Equation $\dot{y} = \frac{ay}{2x}$, or $2x\dot{y} = ay$, by taking the Fluxions again, and putting $\dot{x} = a$ as before, we shall have $2a\ddot{y} + 2x\ddot{y} = a\dot{y}$, or $-\ddot{y} = \frac{a\dot{y}}{2x}$; where the negative sign shews only, that $\ddot{y}$ is to be consider'd rather as a retardation than an acceleration, or an acceleration the contrary way. Now this will give us the following



following Construction. Make x (AD) : y (DG) :: $\frac{1}{2}a$ ($\frac{1}{2}$ AP) : DI = $\ddot{y}$, and the Line DI will therefore represent the Fluxion of DG, or of y , and therefore the second Fluxion of BD, or of y . And if this be done every where upon AE, a Curve IK will be constructed, whose Ordinates will always expound the second Fluxions of the corresponding Ordinates of the Parabola ABC. This Curve likewise will be one of the Hyperbola's, for its Equation is $-\ddot{y} = \frac{ay}{2x} = \frac{a^{\frac{5}{2}}}{4x^{\frac{3}{2}}}$, or $\ddot{y}\ddot{y} = \frac{a^5}{16x^3}$.

Again, from the Equation $-\ddot{y} = \frac{ay}{2x}$, or $-2x\ddot{y} = ay$, by taking the Fluxions we shall have $-2a\ddot{y} - 2x\ddot{\ddot{y}} = a\ddot{y}$, or $-\ddot{\ddot{y}} = \frac{3a\ddot{y}}{2x}$, which will give us this Construction. Make x (AD) : $\ddot{y}$ (DI) :: $\frac{3}{2}a$ ($\frac{3}{2}$ AP) : DL = $\ddot{\ddot{y}}$, and the Line DL will therefore represent the Fluxion of DI, or of $\ddot{y}$, the second Fluxion of DG, or of y , and the third Fluxion of BD, or of y . And if this be done every where upon AE, a Curve LM will be constructed, whose Ordinates will always expound the third Fluxions of the corresponding Ordinates of the Parabola ABC. This Curve will be an Hyperbola, and its Equation will be $-\ddot{\ddot{y}} = \frac{3a\ddot{y}}{2x} = \frac{3a^{\frac{7}{2}}}{8x^{\frac{3}{2}}}$, or $\ddot{\ddot{y}}\ddot{\ddot{y}} = \frac{9a^7}{64x^3}$.

And so we might proceed to construct Curves, the Ordinates of which (in the present Example) would expound or represent the fourth, fifth, and other orders of Fluxions.

We might likewise proceed in a retrograde order, to find the Curves whose Ordinates shall represent the Fluents of any of these Fluxions, when given. As if we had $y = \frac{a^{\frac{3}{2}}}{2x^{\frac{1}{2}}} = \frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}$, or if the Curve GH were given; by taking the Fluents, (as will be taught in the next Problem,) it would be $y = (a^{\frac{1}{2}}x^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}x}{x^{\frac{1}{2}}}) \frac{2xy}{a}$, which will give us this Construction. Make $\frac{1}{2}a$ ($\frac{1}{2}$ AP) : x (AD) :: y (DG) : DB = $\frac{2xy}{a}$, and the Line DB will represent the Fluent of DG, or of y . And if this be done every where upon the Line AE, a Curve AB will be constructed, whose Ordinates will always expound the Fluents of the corresponding Ordinates of the Curve GH. This Curve will be the common Parabola, whose

Parameter is the Line $AP = a$. For its Equation is $y = a^{\frac{1}{2}}x^{\frac{1}{2}}$, or $yy = ax$.

So if we had the Parabola ABC, we might conceive its Ordinates to represent Fluxions, of which the corresponding Ordinates DQ of some other Curve, suppose QR, would represent the Fluents.

To find which Curve, put y for the Fluent of y , y for the Fluent of y , &c. (That is, let, &c. $y, y, y, y, \dot{y}, \ddot{y}, \ddot{y}, \ddot{y}$, &c. be a Series of Terms proceeding both ways indefinitely, of which every succeeding Term represents the Fluxion of the preceding, and *vice versa*; according to a Notation of our Author's, deliver'd elsewhere.) Then

because it is $y = (a^{\frac{1}{2}}x^{\frac{1}{2}} = a^{\frac{1}{2}}x^{\frac{1}{2}} \times \dot{x} =) \frac{\dot{x}x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$, taking the Fluents it

will be $y = \left(\frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = \frac{2a^{\frac{1}{2}}x^{\frac{3}{2}}}{3a} = \right) \frac{2xy}{3a}$; which will give us this Con-

struction. Make $\frac{3}{2}a (\frac{3}{2}AP) : x (AD) :: y (BD) : \frac{2xy}{3a} = y = DQ$, and the Line DQ will represent the Fluent of DB, or of y . And if the same be done at every point of the Line AE, a Curve QR will be form'd, the Ordinates of which will always expound the Fluents of the corresponding Ordinates of the Parabola ABC. This Curve also will be a Parabola, but of a higher order, the Equation of which is $y = \frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}}$, or $yy = \frac{4x^3}{9a}$.

Again, because $y = \left(\frac{2x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} = \frac{2x^{\frac{3}{2}} \times \dot{x}}{3a^{\frac{1}{2}} \times a} = \right) \frac{2\dot{x}x^{\frac{3}{2}}}{3a^{\frac{3}{2}}}$; taking the Fluents it will be $y = \left(\frac{4x^{\frac{5}{2}}}{15a^{\frac{3}{2}}} = \frac{2x^{\frac{3}{2}} \times 2x}{3a^{\frac{1}{2}} \times 5a} = \right) \frac{2xy}{5a}$, which will give us this

Construction. Make $\frac{5}{2}a (\frac{5}{2}AP) : x (AD) :: y (DQ) : \frac{2xy}{5a} = y = DS$, and the Line DS will represent the Fluent of DQ, or of y . And if the same be done at every point of the Line AE, a Curve ST will thereby be form'd, the Ordinates of which will expound the Fluents of the corresponding Ordinates of the Curve QR. This

Curve will be a Parabola, whose Equation is $y = \frac{4x^{\frac{5}{2}}}{15a^{\frac{3}{2}}}$, or $yy =$

$\frac{16x^5}{225a^3}$. And so we might go on as far as we please.

Lastly,

Lastly, if we conceive DB, the common Ordinate of all these Curves, to be any where thus constructed upon AD, that is, to be thus divided in the points S, Q, B, G, I, L, &c. from whence to AP are drawn Ss, Qq, Bb, Gg, Ii, Ll, &c. parallel to AE; and if this Ordinate be farther conceived to move either backwards or forwards upon AE, with an equable Velocity, (represented by $AP = a = \dot{x}$;) and as it describes these Curves, to carry the afore-said Parallels along with it in its motion: Then the points s, q, b, g, i, l, &c. will likewise move in such a manner, in the Line AP, as that the Velocity of each point will be represented by the distance of the next from the point A. Thus the Velocity of s will be represented by Aq, the Velocity of q by Ab, of b by Ag, of g by Ai, of i by Al, &c. Or in other words, Aq will be the Fluxion of As; Ab will be the Fluxion of Aq, or the second Fluxion of As; Ag will be the Fluxion of Ab, or the second Fluxion of Aq, or the third Fluxion of As; Ai will be the Fluxion of Ag, or the second Fluxion of Ab, or the third Fluxion of Aq, or the fourth Fluxion of As; and so on. Now in this instance the several orders of Fluxions, or Velocities, are not only expounded by their Proxies and Representatives, but also are themselves actually exhibited, as far as may be done by Geometrical Figures. And the like obtains wherever else we make a beginning; which sufficiently shews the relative nature of all these orders of Fluxions and Fluents, and that they differ from each other by mere relation only, and in the manner of conceiving. And in general, what has been observed from this Example, may be easily accommodated to any other cases whatsoever.

Or these different orders of Fluents and Fluxions may be thus explain'd abstractedly and Analytically, without the assistance of Curvelines, by the following general Example. Let any constant and known quantity be denoted by a , and let a^m be any given Power or Root of the same. And let x^m be the like Power or Root of the variable and indefinite quantity x . Make $a^m : x^m :: a : y$, or

$$y = \frac{ax^m}{a^m} = a^{1-m}x^m. \text{ Here } y \text{ also will be an indefinite quantity,}$$

which will become known as soon as the value of x is assign'd. Then taking the Fluxions, it will be $\dot{y} = ma^{1-m}x^{m-1}$; and supposing x to flow or increase uniformly, and making its constant Velocity or Fluxion $\dot{x} = a$, it will be $\dot{y} = ma^{1-m}x^{m-1}$. Here if for $a^{1-m}x^m$ we write its value y , it will be $\dot{y} = \frac{may}{x}$, that is, $x : ma :: y : \dot{y}$. So that $\dot{y}$ will be also a known and assignable Quantity,

tity, whenever x (and therefore y) is assign'd. Then taking the Fluxions again, we shall have $\dot{y} = m \times m - 1 a^{2-m} x^{m-2} = m \times m - 1 a^{2-m} x^{m-2}$; or for $ma^{2-m} x^{m-1}$ writing its value $\dot{y}$, it will be $\ddot{y} = \frac{m-1 a \dot{y}}{x}$, that is, $x : m - 1 a :: \dot{y} : \ddot{y}$. So that $\ddot{y}$ will become a known quantity, when x (and therefore y and $\dot{y}$) is assign'd.

Then taking the Fluxions again, we shall have $\ddot{y} = m \times m - 1 \times m - 2 a^{3-m} x^{m-3}$, or $\ddot{y} = \frac{m-2 a \ddot{y}}{x}$, that is, $x : m - 2 a :: \ddot{y} : \ddot{\ddot{y}}$; where also $\ddot{y}$ will be known, when x is given. And taking the

Fluxions again, we shall have $\ddot{\ddot{y}} = m \times m - 1 \times m - 2 \times m - 3 a^{4-m} x^{m-4} = \frac{m-3 a \ddot{\ddot{y}}}{x}$; that is, $x : m - 3 a :: \ddot{\ddot{y}} : \ddot{\ddot{\ddot{y}}}$. So that $\ddot{\ddot{\ddot{y}}}$ will also be known, whenever x is given. And from this Induction we may conclude in general, that if the order of Fluxions be denoted by any integer number n , or if n be put for the number of points over the

Letter y , it will always be $x : m - na :: \dot{y} : \ddot{y}$; or from the Fluxion of any order being given, the Fluxion of the next immediate order may be hence found.

Or we may thus invert the proportion $m - na : x :: \dot{y} : \ddot{y}$, and then from the Fluxion given, we shall find its next immediate Fluent. As if $n = 2$, 'tis $m - 2a : x :: \dot{y} : \ddot{y}$. If $n = 1$, 'tis $m - 1a : x :: \ddot{y} : \dot{y}$. If $n = 0$, 'tis $ma : x :: \dot{y} : y$. And observing the same analogy, if $n = -1$, 'tis $m + 1a : x :: y : \dot{y}$; where y is put for the Fluent of y , or for y with a negative point.

And here because $y = a^{1-m} x^m$, it will be $m + 1a : x :: a^{1-m} x^m : y$, or $y = \frac{a^{1-m} x^{m+1}}{m+1a} = \frac{x^{m+1}}{m+1a}$: which also may thus appear. Be-

cause $y = (a^{1-m} x^m = \frac{a^{1-m} x^m}{a} = \frac{\dot{x} x^m}{a})$, taking the Fluents, (see the next Problem,) it will be $y = \frac{x^{m+1}}{m+1a}$. Again, if we make $n = -2$,

'tis $m + 2a : x :: \dot{y} : y$, or $y = \frac{xy}{m+2a} = \frac{x^{m+2}}{m+1 \times m+2a}$. For
M m because

because $y = \frac{x^{m+1}}{m+1a} \times \frac{\dot{x}}{a} = \frac{\dot{x}x^{m+1}}{m+1a}$, taking the Fluents it will be

// $y = \frac{x^{m+2}}{m+1 \times m+2a}$. Again, if we make $n = -3$, 'tis $m+3a$:

// $x :: y : y$, or $y = \frac{xy}{m+3a} = \frac{x^{m+3}}{m+1 \times m+2 \times m+3a}$. And so for

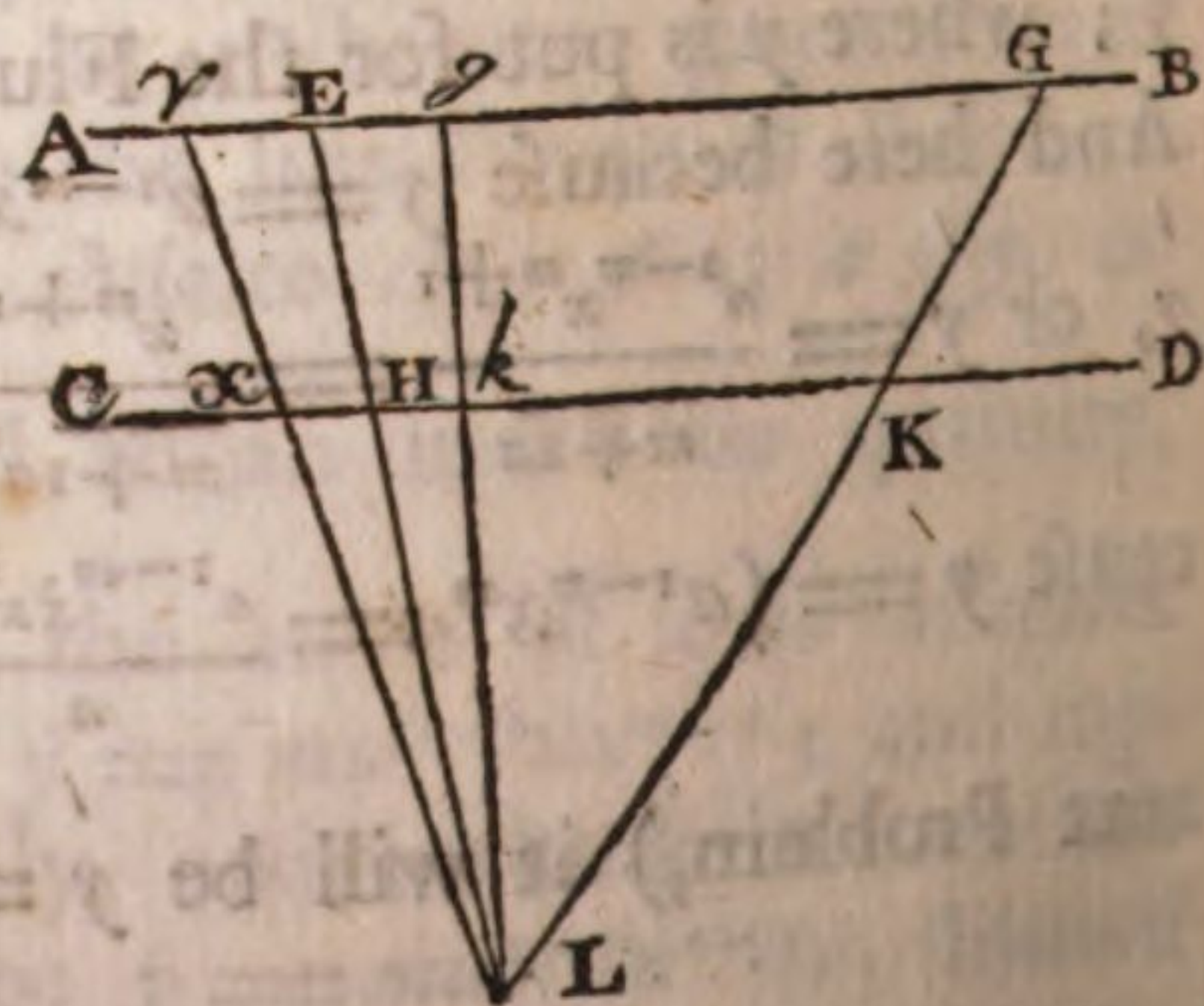
all other superior orders of Fluents.

And this may suffice in general, to shew the comparative nature and properties of these several orders of Fluxions and Fluents, and to teach the operations by which they are produced, or to find their respective fluxional Equations. As to the uses they may be apply'd to, when found, that will come more properly to be consider'd in another place.

SECT. III. *The Geometrical and Mechanical Elements of Fluxions.*

THE foregoing Principles of the Doctrine of Fluxions being chiefly abstracted and Analytical, I shall here endeavour, after a general manner, to shew something analogous to them in Geometry and Mechanicks; by which they may become, not only the object of the Understanding, and of the Imagination, (which will only prove their possible existence,) but even of Sense too, by making them actually to exist in a visible and sensible form. For it is now become necessary to exhibit them all manner of ways, in order to give a satisfactory proof, that they have indeed any real existence at all.

And first, by way of preparation, it will be convenient to consider uniform and equable motions, as also such as are alike inequable. Let the right Line AB be described by the equable motion of a point, which is now at E, and will presently be at G. Also let the Line CD, parallel to the former, be described by the equable motion of a point, which is in H and K, at the same times as the former is in E and G. Then will EG and HK be contemporaneous Lines, and therefore will be proportional to the



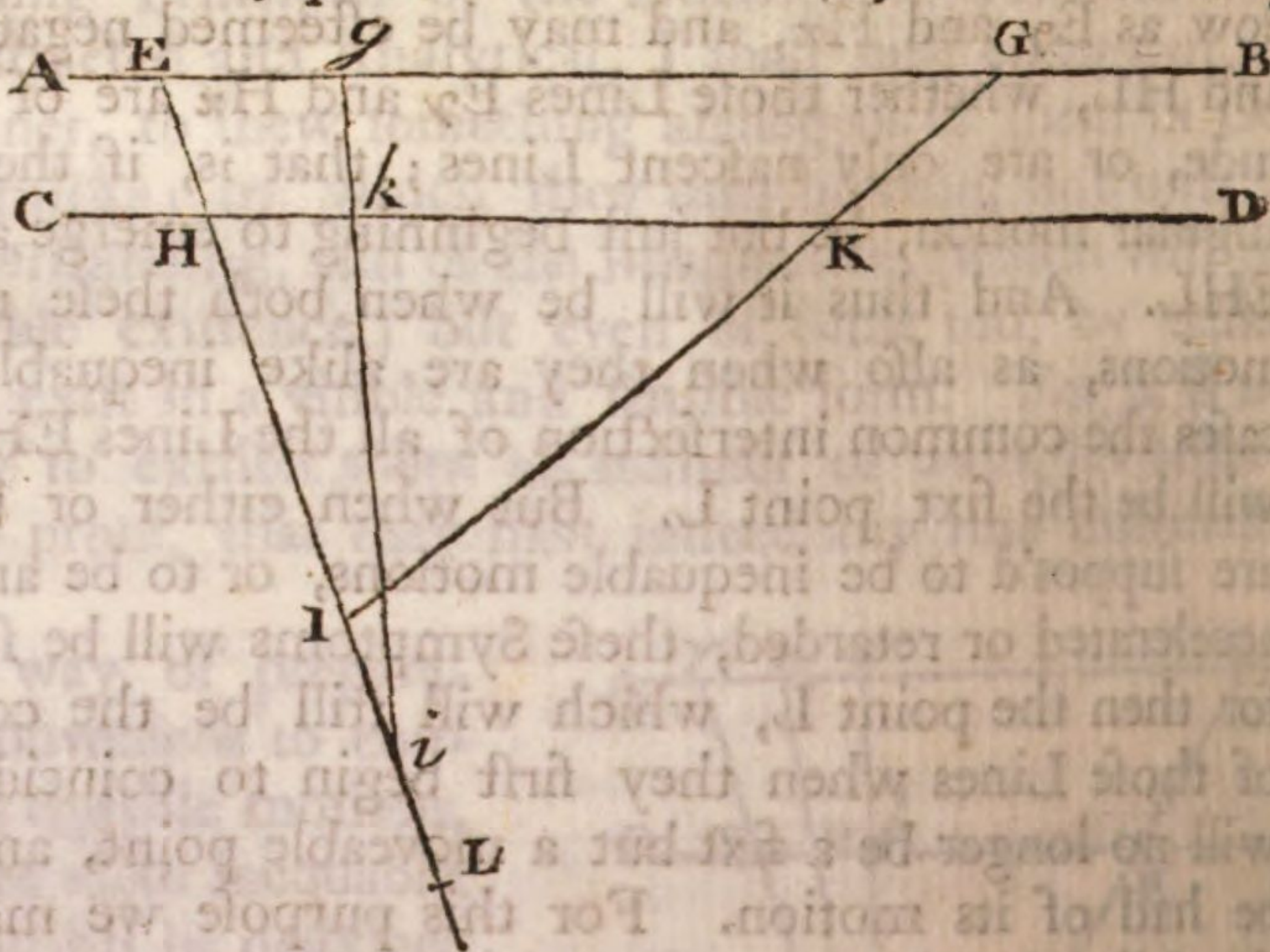
the Velocity of each moving point respectively. Draw the indefinite Lines EH and GK , meeting in L ; then because of like Triangles ELG and HLK , the Velocities of the points E and H , which were before as EG and HK , will be now as EL and HL . Let the describing points G and K be conceived to move back again, with the same Velocities, towards A and C , and before they approach to E and H let them be found in g and k , at any small distance from E and H , and draw gk , which will pass through L ; then still their Velocities will be in the ratio of Eg and Hk , be those Lines ever so little, that is, in the ratio of EL and HL . Let the moving points g and k continue to move till they coincide with E and H ; in which case the decreasing Lines Eg and Hk will pass through all possible magnitudes that are less and less, and will finally become vanishing Lines. For they must intirely vanish at the same moment, when the points g and k shall coincide with E and H . In all which states and circumstances they will still retain the ratio of EL to HL , with which at last they will finally vanish. Let those points still continue to move, after they have coincided with E and H , and let them be found again at the same time in γ and κ , at any distance beyond E and H . Still the Velocities, which are now as $E\gamma$ and $H\kappa$, and may be esteemed negative, will be as EL and HL , whether those Lines $E\gamma$ and $H\kappa$ are of any finite magnitude, or are only nascent Lines; that is, if the Line $\gamma\kappa L$, by its angular motion, be but just beginning to emerge and divaricate from EHL . And thus it will be when both these motions are equable motions, as also when they are alike inequable; in both which cases the common intersection of all the Lines EHL , GKL , gkL , &c. will be the fixt point L . But when either or both these motions are suppos'd to be inequable motions, or to be any how continually accelerated or retarded, these Symptoms will be something different; for then the point L , which will still be the common intersection of those Lines when they first begin to coincide, or to divaricate, will no longer be a fixt but a moveable point, and an account must be had of its motion. For this purpose we may have recourse to the following Lemma.

Let AB be an indefinite and fixt right Line, along which another indefinite but moveable right Line DE may be conceived to move or roll in such a manner, as to have both a progressive motion, as also an angular motion about a moveable Center C . That is, the common intersection C of the two Lines AB and DE may be supposed to move with any progressive motion from A towards B , while at the

same time the moveable Line DE revolves about the same point C, with any angular motion. Then as the Angle ACD continually decreases, and at last vanishes when the two Lines ACB and DCE coincide; yet even then the point of intersection C, (as it may be still call'd,) will not be lost and annihilated, but will appear again, as soon as the Lines begin to divaricate, or to separate from each other. That is, if C be the point of intersection before the coincidence, and *c* the point of intersection after the coincidence, when the Line *dce* shall again emerge out of AB; there will be some intermediate point L, in which C and *c* were united in the same point, at the moment of coincidence. This point, for distinction-sake, may be call'd the *Node*, or *the point of no divarication*. Now to apply this to inequable Motions:



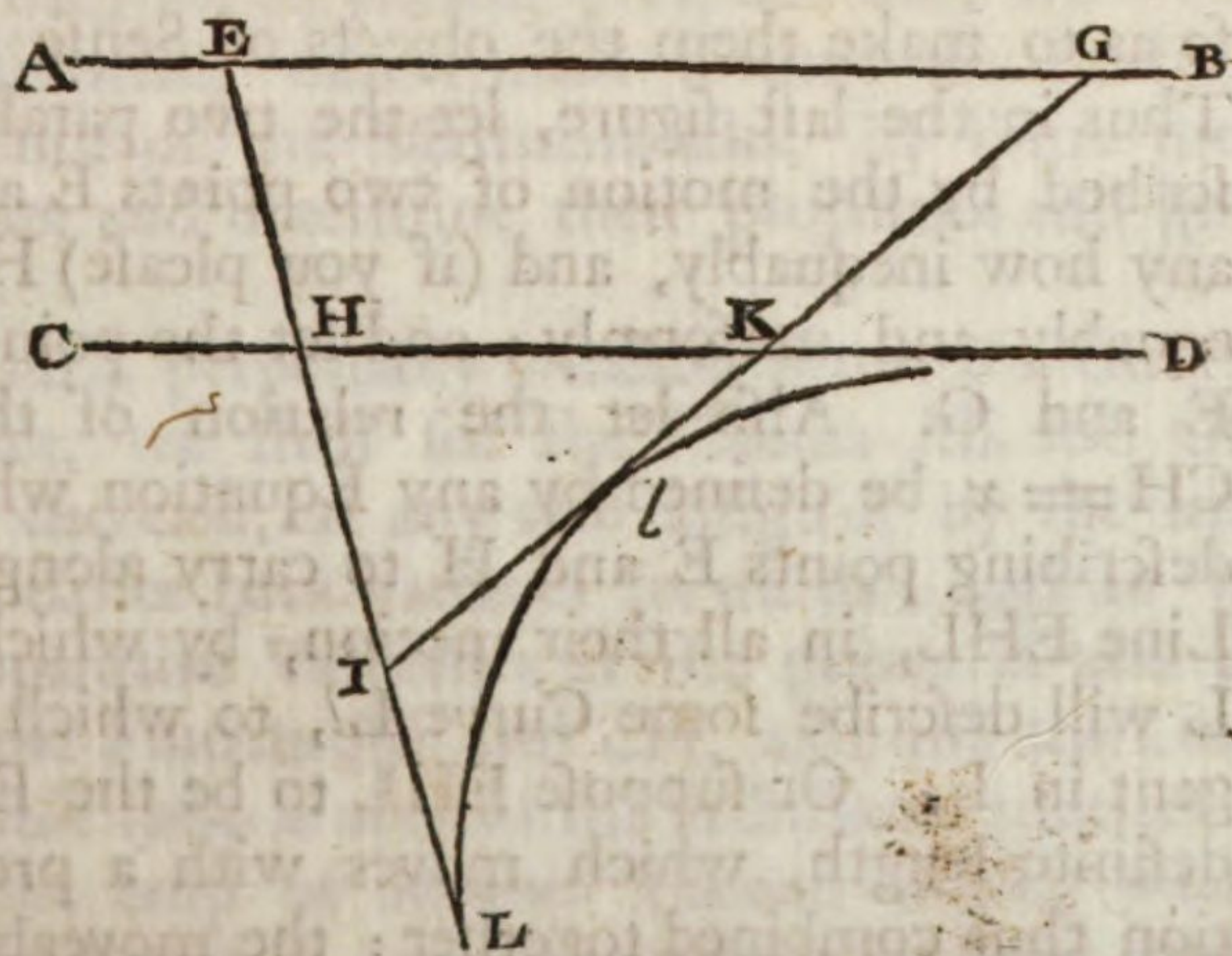
Let the Line AB be described by the continually accelerated motion of a point, which is now in E, and will be presently found in G. Also let the Line CD, parallel to the former, be described by the equable motion of a point, which is found in H and K, at the same times as the other point is in E and G. Then will EG and HK be contemporaneous Lines; and producing EH and GK till they meet in I, those contemporaneous Lines will be as EI and HI respectively. Let the describing points G and K be conceived to move back again towards A and C, each with the same degrees of Velocity, in every point of their motion, as they had before acquired; and let them arrive at the same time at *g* and *k*, at some small distance from E and H, and draw *gki* meeting EH in *i*. Then Eg and Hk, being contemporary Lines also, and very little by supposition, they will be nearly as the Velocities



locities at g and k , that is, at E and H ; which contemporary Lines will be now as Ei and Hi . Let the points g and k continue their motion till they coincide with E and H , or let the Line GKI or gki continue its progressive and angular motion in this manner, till it coincides with EHL , and let L be the Node, or point of no divarication, as in the foregoing Lemma. Then will the last ratio of the vanishing Lines Eg and Hk , which is the ratio of the Velocities at E and H , be as EL and HL respectively.

Hence we have this Corollary. If the point E (in the foregoing figure,) be suppos'd to move from A towards B , with a Velocity any how accelerated, and at the same time the point H moves from C towards D with an equable Velocity, (or inequable, if you please;) those Velocities in E and H will be respectively as the Lines EL and HL , which point L is to be found, by supposing the contemporary Lines EG and HK continually to diminish, and finally to vanish. Or by supposing the moveable indefinite Line GKI to move with a progressive and angular motion, in such manner, as that EG and HK shall always be contemporary Lines, till at last GKI shall coincide with the Line EHL , at which time it will determine the Node L , or the point of no divarication. So that if the Lines AE and CH represent two Fluents, any how related, their Velocities of description at E and H , or their respective Fluxions, will be in the ratio of EL and HL .

And hence it will follow also, that the Locus of the moveable point or Node L , that is, of all the points of no divarication, will be some Curve-line Ll , to which the Lines EHL and GKI will always be Tangents in L and l . And the nature of this Curve Ll may be determined by the given relation of the Fluents or Lines AE and CH ; and *vice versâ*. Or however the relation of its intercepted Tangents EL and HL may be determined in all cases; that is, the ratio of the Fluxions of the given Fluents.



For

For illustration-sake, let us apply this to an Example. Make the Fluents $AE = y$ and $CH = x$, and let the relation of these be always express'd by this Equation $y = x^n$. Make the contemporary Lines $EG = Y$ and $HK = X$; and because AE and CH are contemporary by supposition, we shall have the whole Lines AG and CK contemporary also, and thence the Equation $y + Y = x + X$. This by our Author's Binomial Theorem will produce $y + Y = x^n + nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. which (because $y = x^n$) will become $Y = nx^{n-1}X + n \times \frac{n-1}{2}x^{n-2}X^2$, &c. or in an Analogy, $X : Y :: 1 : nx^{n-1} + n \times \frac{n-1}{2}x^{n-2}X$, &c. which will be the general relation of the contemporary Lines or Increments EG and HK . Now let us suppose the indefinite Line GKI , which limits these contemporary Lines, to return back by a progressive and angular motion, so as always to intercept contemporary Lines EG and HK , and finally to coincide with EHL , and by that means to determine the Node L ; that is, we may suppose $EG = Y$ and $HK = X$, to diminish *in infinitum*, and to become vanishing Lines, in which case we shall have $X : Y :: 1 : nx^{n-1}$. But then it will be likewise $X : Y :: HK : EG :: HL : EL :: \dot{x} : \dot{y}$, or $1 : nx^{n-1} :: \dot{x} : \dot{y}$, or $\dot{y} = n\dot{x}x^{n-1}$.

And hence we may have an expedient for exhibiting Fluxions and Fluents Geometrically and Mechanically, in all circumstances, so as to make them the objects of Sense and ocular Demonstration. Thus in the last figure, let the two parallel lines AB and CD be described by the motion of two points E and H , of which E moves any how inequably, and (if you please) H may be suppos'd to move equably and uniformly; and let the points H and K correspond to E and G . Also let the relation of the Fluents $AE = y$ and $CH = x$ be defined by any Equation whatever. Suppose now the describing points E and H to carry along with them the indefinite Line EHL , in all their motion, by which means the point or Node L will describe some Curve LI , to which EL will always be a Tangent in L . Or suppose EHL to be the Edge of a Ruler, of an indefinite length, which moves with a progressive and angular motion thus combined together; the moveable point or Node L in this Line, which will have the least angular motion, and which is always the point of no divarication, will describe the Curve, and the Line or Edge itself will be a Tangent to it in L . Then will the segments EL and HL be proportional to the Velocity of the points E and H respectively; or will exhibit the ratio of the Fluxions $\dot{y}$ and $\dot{x}$, belonging to the Fluents $AE = y$ and $CH = x$.

Or if we suppose the Curve Ll to be given, or already constructed, we may conceive the indefinite Line $EHIL$ to revolve or roll about it, and by continually applying itself to it, as a Tangent, to move from the situation $EHIL$ to GK/I . Then will AE and CH be the Fluents, the sensible velocities of the describing points E and H will be their Fluxions, and the intercepted Tangents EL and HL will be the rectilinear measures of those Fluxions or Velocities. Or it may be represented thus: If Ll be any rigid obstacle in form of a Curve, about which a flexible Line, or Thread, is conceived to be wound, part of which is stretch'd out into a right Line LE , which will therefore touch the Curve in L ; if the Thread be conceived to be farther wound about the Curve, till it comes into the situation L/KG ; by this motion it will exhibit, even to the Eye, the same increasing Fluents as before, their Velocities of increase, or their Fluxions, as also the Tangents or rectilinear representatives of those Fluxions. And the same may be done by unwinding the Thread, in the manner of an Evolute. Or instead of the Thread we may make use of a Ruler, by applying its Edge continually to the curved Obstacle Ll , and making it any how revolve about the moveable point of Contact L or l . In all which manners the Fluents, Fluxions, and their rectilinear measures, will be sensibly and mechanically exhibited, and therefore they must be allowed to have a place *in rerum naturâ*. And if they are in nature, even tho' they were but barely possible and conceivable, much more if they are sensible and visible, it is the province of the Mathematicks, by some method or other, to investigate and determine their properties and proportions.

Or as by one Thread EHL , perpetually winding about the curved obstacle Ll , of a due figure, we shall see the Fluents AE and CH continually to increase or decrease, at any rate assign'd, by the motion of the Thread EHL either backwards or forwards; and as we shall thereby see the comparative Velocities of the points E and H ; that is, the Fluxions of the Fluents AE and CH , and also the Lines EL and HL , whose variable ratio is always the rectilinear measure of those Fluxions: So by the help of another Thread GK/L , winding about the obstacle in its part lL , and then stretching out into a right Line or Tangent $/KG$, and made to move backwards or forwards, as before; if the first Thread be at rest in any given situation EHL , we may see the second Thread describe the contemporary Lines or Increments EG and HK , by which the Fluents AE and CH are continually increased; and if GK/l is made to approach

proach towards EHL, we may see those contemporary Lines continually to diminish, and their ratio continually approaching towards the ratio of EL to HL; and continuing the motion, we may presently see those two Lines actually to coincide, or to unite as one Line, and then we may see the contemporary Lines actually to vanish at the same time, and their ultimate ratio actually to become that of EL to HL. And if the motion be still continued, we shall see the Line GK/ to emerge again out of EHL, and begin to describe other contemporary Lines, whose nascent proportion will be that of EL to HL. And so we may go on till the Fluents are exhausted. All these particulars may be thus easily made the objects of sight, or of Ocular Demonstration.

This may still be added, that as we have here exhibited and represented first Fluxions geometrically and mechanically, we may do the same thing, *mutatis mutandis*, by any higher orders of Fluxions. Thus if we conceive a second figure, in which the Fluential Lines shall increase after the rate of the ratio of the intercepted Tangents (or the Fluxions) of the first figure; then its intercepted Tangents will expound the ratio of the second Fluxions of the Fluents in the first figure. Also if we conceive a third figure, in which the Fluential Lines shall increase after the rate of the intercepted Tangents of the second figure; then its intercepted Tangents will expound the third Fluxions of the Fluents in the first figure. And so on as far as we please. This is a necessary consequence from the relative nature of these several orders of Fluxions, which has been shewn before.

And farther to shew the universality of this Speculation, and how well it is accommodated to explain and represent all the circumstances of Fluxions and Fluents; we may here take notice, that it may be also adapted to those cases, in which there are more than two Fluents, which have a mutual relation to each other, express'd by one or more Equations. For we need but introduce a third parallel Line, and suppose it to be described by a third point any how moving, and that any two of these describing points carry an indefinite Line along with them, which by revolving as a Tangent, describes the Curve whose Tangents every where determine the Fluxions. As also that any other two of those three points are connected by another indefinite Line, which by revolving in like manner describes another such Curve. And so there may be four or more parallel Lines. All but one of these Curves may be assumed at pleasure, when they are not given by the state of the Question. Or Analytically,

tically, so many Equations may be assumed, except one, (if not given by the Problem,) as is the number of the Fluents concern'd.

But lastly, I believe it may not be difficult to give a pretty good notion of Fluents and Fluxions, even to such Persons as are not much versed in Mathematical Speculations, if they are willing to be inform'd, and have but a tolerable readiness of apprehension. This I shall here attempt to perform, in a familiar way, by the instance of a Fowler, who is aiming to shoot two Birds at once, as is represented in the Frontispiece. Let us suppose the right Line AB to be parallel to the Horizon, or level with the Ground, in which a Bird is now flying at G, which was lately at F, and a little before at E. And let this Bird be conceived to fly, not with an equable or uniform swiftness, but with a swiftness that always increases, (or with a Velocity that is continually accelerated,) according to some known rate. Let there also be another right Line CD, parallel to the former, at the same or any other convenient distance from the Ground, in which another Bird is now flying at K, which was lately at I, and a little before at H; just at the same points of time as the first Bird was at G, F, E, respectively. But to fix our Ideas, and to make our Conceptions the more simple and easy, let us imagine this second Bird to fly equably, or always to describe equal parts of the Line CD in equal times. Then may the equable Velocity of this Bird be used as a known measure, or standard, to which we may always compare the inequable Velocity of the first Bird. Let us now suppose the right Line EH to be drawn, and continued to the point L, so that the proportion (or ratio) of the two Lines EL and HL may be the same as that of the Velocities of the two Birds, when they were at E and H respectively. And let us farther suppose, that the Eye of a Fowler was at the same time at the point L, and that he directed his Gun, or Fowling-piece, according to the right Line LHE, in hopes to shoot both the Birds at once. But not thinking himself then to be sufficiently near, he forbears to discharge his Piece, but still pointing it at the two Birds, he continually advances towards them according to the direction of his Piece, till his Eye is presently at M, and the Birds at the same time in F and I, in the same right Line FIM. And not being yet near enough, we may suppose him to advance farther in the same manner, his Piece being always directed or level'd at the two Birds, while he himself walks forward according to the direction of his Piece, till his Eye is now at N, and the Birds in the same right Line with his Eye, at K and G. The Path of his Eye, described by this

N n

double

double motion, (or compounded of a progressive and angular motion,) will be some Curve-line LMN, in the same Plain as the rest of the figure, which will have this property, that the proportion of the distances of his Eye from each Bird, will be the same every where as that of their respective Velocities. That is, when his Eye was at L, and the Birds at E and H, their Velocities were then as EL and HL, by the Construction. And when his Eye was at M, and the Birds at F and I, their Velocities were in the same proportion as the Lines FM and IM, by the nature of the Curve LMN. And when his Eye is at N, and the Birds at G and K, their Velocities are in the proportion of GN to KN, by the nature of the same Curve. And so universally, of all other situations. So that the Ratio of those two Lines will always be the sensible measure of the ratio of those two sensible Velocities. Now if these Velocities, or the swiftnesses of the flight of the two Birds in this instance, are call'd *Fluxions*; then the Lines described by the Birds in the same time, may be call'd their *contemporaneous Fluents*; and all instances whatever of Fluents and Fluxions, may be reduced to this Example, and may be illustrated by it.

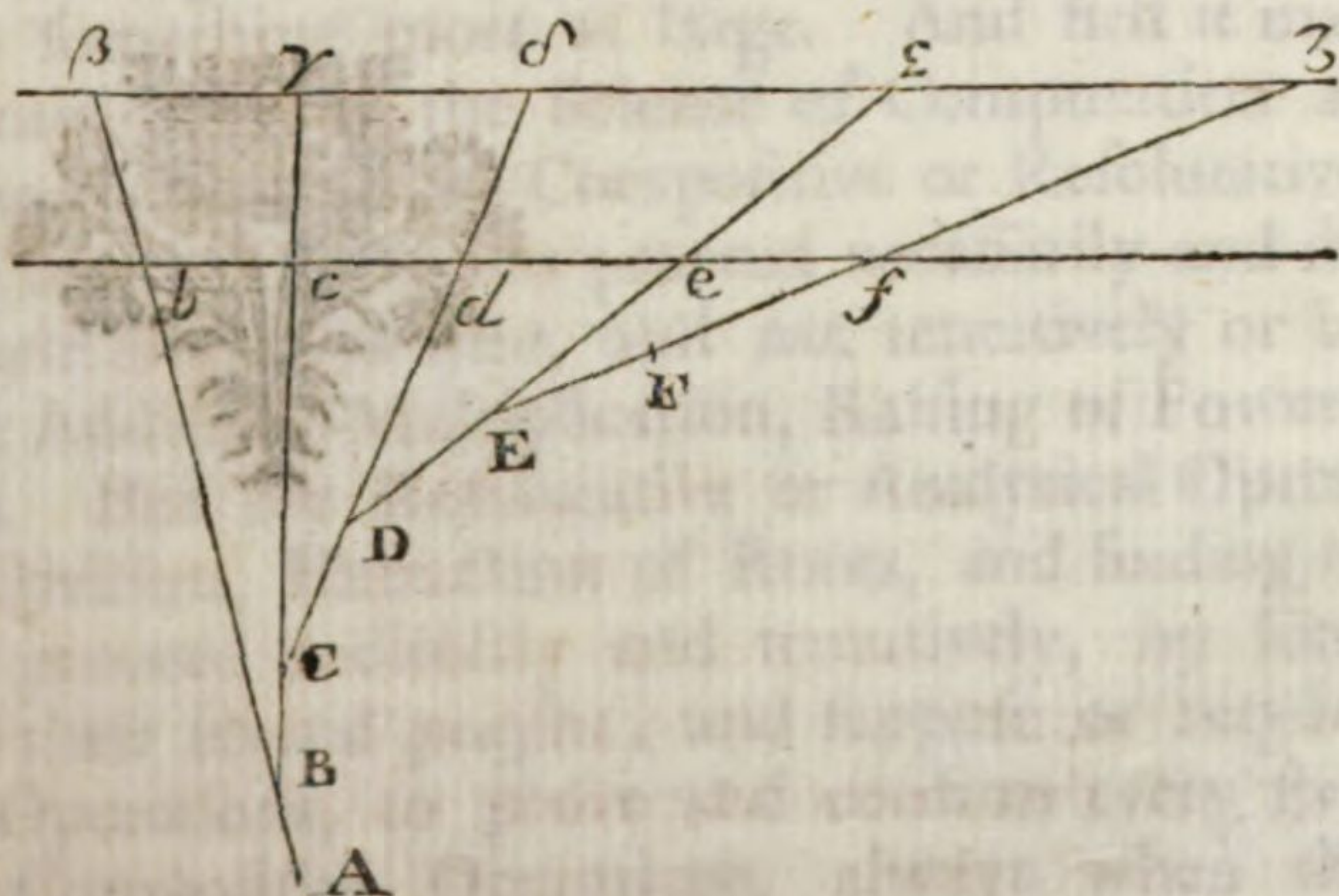
And thus I would endeavour to give some notion of Fluents and Fluxions, to Persons not much conversant in the Mathematicks; but such as had acquired some skill in these Sciences, I would thus proceed farther to instruct, and to apply what has been now deliver'd. The contemporaneous Fluents being $EF = y$, and $HI = x$, and their rate of flowing or increasing being suppos'd to be given or known; their relation may always be express'd by an Equation, which will be compos'd of the variable quantities x and y , together with any known quantities. And that Equation will have this property, because of those variable quantities, that as FG and IK, EG and HK, and infinite others, are also contemporaneous Fluents; it will indifferently exhibit the relation of those Lines also, as well as of EF and HI; or they may be substituted in the Equation, instead of x and y . And hence we may derive a Method for determining the Velocities themselves, or for finding Lines proportional to them. For making $FG = Y$, and $IK = X$; in the given Equation I may substitute $y + Y$ instead of y , and $x + X$ instead of x , by which I shall obtain an Equation, which in all circumstances will exhibit the relation of those Quantities or Increments. Now it may be plainly perceived, that if the Line MIF is conceived continually to approach nearer and nearer to the Line NKG, (as just now, in the instance of the Fowler,) till it finally coincides with it; the Lines $FG = Y$,
and

and $IK = X$, will continually decrease, and by decreasing will approach nearer and nearer to the Ratio of the Velocities at G and K, and will finally vanish at the same time, and in the proportion of those Velocities, that is, in the Ratio of GN to KN. Consequently in the Equation now form'd, if we suppose Y and X to decrease continually, and at last to vanish, that we may obtain their ultimate Ratio; we shall thereby obtain the Ratio of GN to KN. But when Y and X vanish, or when the point F coincides with G, and I with H, then it will be $EG = y$, and $HK = x$; so that we shall have $y : x :: GM : KN$. And hence we shall obtain a Fluxional Equation, which will always exhibit the relation of the Fluxions, or Velocities, belonging to the given Algebraical or Fluent Equation.

Thus, for Example, if $EF = y$, and $HI = x$, and the indefinite Lines y and x are supposed to increase at such a rate, as that their relation may always be express'd by this Equation $x^3 - ax^2 + axy - y^3 = 0$; then making $FG = Y$, and $IK = X$, by substituting $y + Y$ for y , and $x + X$ for x , and reducing the Equation that will arise, (see before, pag. 255.) we shall have $3x^2X + 3xX^2 + X^3 - 2axX - aX^2 + axY + aXY + aXY - 3y^2Y - 3yY^2 - Y^3 = 0$, which may be thus express'd in an Analogy, $Y : X :: 3x^2 - 2ax + ay + 3xX + X^2 - aX : 3y^2 - ax - aX + 3yY + Y^2$. This Analogy, when Y and X are vanishing quantities, or their ultimate Ratio, will become $Y : X :: 3x^2 - 2ax + ay : 3y^2 - ax$. And because it is then $Y : X :: GN : KN :: y : x$, it will be $y : x :: 3x^2 - 2ax + ay : 3y^2 - ax$. Which gives the proportion of the Fluxions. And the like in all other cases. Q. E. I.

We might also lay a foundation for these Speculations in the following manner. Let

ABCDEF, &c. be the Periphery of a Polygon, or any part of it, and let the Sides AB, BC, CD, DE, &c. be of any magnitude whatever. In the same Plane, and at any distance, draw the two parallel Lines $\beta\zeta$, and bf , to which continue the right Lines $ABb\beta$, $BCc\gamma$, $CDd\delta$, $DEee$, &c. meeting the parallels as in the figure. Now if we suppose



pose two moving points, or bodies, to be at β and b , and to move in the same time to γ and c , with any equable Velocities; those Velocities will be to each other as $\beta\gamma$ and bc , that is, because of the parallels, as βB and bB . Let them set out again from γ and c , and arrive at the same time at δ and d , with any equable Velocities; those Velocities will be as $\gamma\delta$ and cd , that is, as γC and cC . Let them depart again from δ and d , and arrive in the same time at e and e , with any equable Velocities; those Velocities will be as δe and de , that is, as δD and dD . And it will be the same thing every where, how many soever, and how small soever, the Sides of the Polygon may be. Let their number be increased, and their magnitude be diminish'd *in infinitum*, and then the Periphery of the Polygon will continually approach towards a Curve-line, to which the Lines $ABb\beta$, $BCc\gamma$, $CDd\delta$, &c. will become Tangents; as also the Motions may be conceived to degenerate into such as are accelerated or retarded continually. Then in any two points, suppose δ and d , where the describing points are found at the same time, their Velocities (or Fluxions) will be as the Segments of the respective Tangents δD and dD ; and the Lines $\beta\delta$ and bd , intercepted by any two Tangents δD and βB , will be the contemporaneous Lines, or Fluents. Now from the nature of the Curve being given, or from the property of its Tangents, the contemporaneous Lines may be found, or the relation of the Fluents. And *vice versâ*, from the Rate of flowing being given, the corresponding Curve may be found.





ANNOTATIONS on Prob. 2.

O R,

The Relation of the Fluxions being given, to find the Relation of the Fluents.

SECT. I. *A particular Solution ; with a preparation for the general Solution, by which it is distributed into three Cases.*

I, 2. **W**E are now come to the Solution of the Author's second fundamental Problem, borrow'd from the Science of Rational Mechanics : Which is, from the Velocities of the Motion at all times given, to find the quantities of the Spaces described ; or to find the Fluents from the given Fluxions. In discussing which important Problem, there will be occasion to expatiate something more at large. And first it may not be amiss to take notice, that in the Science of Computation all the Operations are of two kinds, either Compositive or Resolutive. The Compositive or Synthetic Operations proceed necessarily and directly, in computing their several *quæsitæ*, and not tentatively or by way of tryal. Such are Addition, Multiplication, Raising of Powers, and taking of Fluxions. But the Resolutive or Analytical Operations, as Subtraction, Division, Extraction of Roots, and finding of Fluents, are forced to proceed indirectly and tentatively, by long deductions, to arrive at their several *quæsitæ* ; and suppose or require the contrary Synthetic Operations, to prove and confirm every step of the Process. The Compositive Operations, always when the *data* are finite and terminated, and often when they are interminate

or infinite, will produce finite conclusions; whereas very often in the Resolutive Operations, tho' the *data* are in finite Terms, yet the *quaesita* cannot be obtain'd without an infinite Series of Terms. Of this we shall see frequent Instances in the subsequent Operation, of returning to the Fluents from the Fluxions given.

The Author's particular Solution of this Problem extends to such cases only, wherein the Fluxional Equation proposed either has been, or at least might have been, derived from some finite Algebraical Equation, which is now required. Here all the necessary Terms being present, and no more than what are necessary, it will not be difficult, by a Process just contrary to the former, to return back again to the original Equation. But it will most commonly happen, either if we assume a Fluxional Equation at pleasure, or if we arrive at one as the result of some Calculation, that such an Equation is to be resolved, as could not be derived from any previous finite Algebraical Equation, but will have Terms either redundant or deficient; and consequently the Algebraic Equation required, or its Root, must be had by Approximation only, or by an infinite Series. In all which cases we must have recourse to the general Solution of this Problem, which we shall find afterwards.

The Precepts for this particular Solution are these. (1.) All such Terms of the given Equation as are multiply'd (suppose) by $\dot{x}$, must be disposed according to the Powers of x , or must be made a Number belonging to the Arithmetical Scale whose Root is x . (2.) Then they must be divided by $\dot{x}$, and multiply'd by x ; or $\dot{x}$ must be changed into x , by expunging the point. (3.) And lastly, the Terms must be severally divided by the Progression of the Indices of the Powers of x , or by some other Arithmetical Progression, as need shall require. And the same things must be repeated for every one of the flowing quantities in the given Equation.

Thus in the Equation $3\dot{x}x^2 - 2a\dot{x}x + a\dot{y}y - 3\dot{y}y^2 + a\dot{y}x = 0$, the Terms $3\dot{x}x^2 - 2a\dot{x}x + a\dot{y}y$ by expunging the points become $3x^3 - 2ax^2 + axy$, which divided by the Progression of the Indices 3, 2, 1, respectively, will give $x^3 - ax^2 + axy$. Also the Terms $-3\dot{y}y^2 + a\dot{y}x$ by expunging the points become $-3y^3 + ayx$, which divided by the Progression of the Indices 3, 2, 1, respectively, will give $-y^3 + ayx$. The aggregate of these, neglecting the redundant Term axy , is $x^3 - ax^2 - y^3 + ayx = 0$, the Equation required. Where it must be noted, that every Term, which occurs more than once, must be accounted a redundant Term.

So

So if the proposed Equation were $\overline{m + 3y\dot{x}x^3} - \overline{m + 2ay\dot{x}x^2} + \overline{m + 1ay^2\dot{x}x} - \overline{my^4\dot{x}} - \overline{n + 3xy\dot{y}^3} + \overline{n + 1ax^2y\dot{y}} + \overline{nx^4\dot{y}} - \overline{nax^3\dot{y}} = 0$, whatever values the general Numbers m and n may acquire; if those Terms in which $\dot{x}$ is found are reduced to the Scale whose Root is x , they will stand thus: $\overline{m + 3y\dot{x}x^3} - \overline{m + 2ay\dot{x}x^2} + \overline{m + 1ay^2\dot{x}x} - \overline{my^4\dot{x}}$; or expunging the points they will become $\overline{m + 3yx^4} - \overline{m + 2ayx^3} + \overline{m + 1ay^2x^2} - \overline{my^4x}$. These being divided respectively by the Arithmetical Progression $m + 3$, $m + 2$, $m + 1$, m , will give the Terms $\overline{yx^4} - \overline{ayx^3} + \overline{ay^2x^2} - \overline{y^4x}$. Also the Terms in which $\dot{y}$ is found; being reduced to the Scale whose Root is y , will stand thus: $\overline{-n + 3xy\dot{y}^3} + \overline{n + 1ax^2y\dot{y}} + \overline{nx^4\dot{y}} - \overline{nax^3\dot{y}}$

or expunging the points they will become $\overline{-n + 3xy^4} + \overline{n + 1ax^2y^2} + \overline{nx^4y} - \overline{nax^3y}$. These being divided respectively by the Arithmetical Pro-

gression $n + 3$, $n + 2$, $n + 1$, n , will give the Terms $\overline{-xy^4} + \overline{ax^2y^2} + \overline{x^4y} - \overline{ax^3y}$. But these Terms, being the same as the former, must all be consider'd as redundant, and therefore are to be rejected. So that $\overline{yx^4} - \overline{ayx^3} + \overline{ay^2x^2} - \overline{y^4x} = 0$, or dividing by yx , the Equation $\overline{x^3} - \overline{ax^2} + \overline{ayx} - \overline{y^3} = 0$ will arise as before.

Thus if we had this Fluxional Equation $\overline{may\dot{x}x^{-1}} - \overline{m + 2\dot{x}x} - \overline{nx^2y\dot{y}^{-1}} + \overline{n + 1ay} = 0$, to find the Fluential Equation to which it belongs; the Terms $\overline{may\dot{x}x^{-1}} * \overline{-m + 2\dot{x}x}$, by expunging the points, and dividing by the Terms of the Progression m , $m + 1$, $m + 2$, will give the Terms $\overline{ay} - \overline{x^2}$. Also the Terms $\overline{-nx^2y\dot{y}^{-1}} + \overline{n + 1ay}$, by expunging the points, and dividing by n , $n + 1$, will give the Terms $\overline{-x^2} + \overline{ay}$. Now as these are the same as the former, they are to be esteem'd as redundant, and the Equation required will be $\overline{ay} - \overline{x^2} = 0$. And when the given Fluxional Equation is a general one, and adapted to all the forms of the Fluential Equation, as is the case of the two last Examples; then all the Terms arising from the second Operation will be always redundant, so that it will be sufficient to make only one Operation.

Thus if the given Equation were $\overline{4y\dot{y}^2} + \overline{z^3y\dot{y}^{-1}} + \overline{2y\dot{x}x} - \overline{3z\dot{z}^2} + \overline{6yz\dot{z}} - \overline{2cy\dot{z}} = 0$, in which there are found three flowing quantities; the only Term in which $\dot{x}$ is found is $\overline{2y\dot{x}x}$, in which expunging the point, and then dividing by the Index 2, it will become $\overline{yx^2}$. Then the Terms in which $\dot{y}$ is found are $\overline{4y\dot{y}^2} + \overline{z^3y\dot{y}^{-1}}$, which expunging the points become $\overline{4y^3} * * \overline{+ z^3}$, and dividing
by

by the Progression 2, 1, 0, — 1, give the Terms $2y^3 - z^3$. Lastly the Terms in which z is found are $-3\dot{z}z^2 + 6y\dot{z}z - 2cy\dot{z}$, which expunging the points become $-3z^3 + 6yz^2 - 2cyz$, and dividing by the Progression 3, 2, 1, give the Terms $-z^3 + 3yz^2 - 2cyz$. Now if we collect these Terms, and omit the redundant Term $-z^3$, we shall have $yx^2 + 2y^3 - z^3 + 3yz^2 - 2cyz = 0$ for the Equation required.

3, 4. But these deductions are not to be too much rely'd upon, till they are verify'd by a proof; and we have here a sure method of proof, whether we have proceeded rightly or not, in returning from the relation of the Fluxions to the relation of the Fluents. For every resolutative Operation should be proved by its contrary compositive Operation. So if the Fluxional Equation $\dot{x}x - \dot{x}y - x\dot{y} + ay = 0$ were given, to return to the Equation involving the Fluents; by the foregoing Rule we shall first have the Terms $\dot{x}x - \dot{x}y$, which by expunging the points will become $x^2 - xy$, and dividing by the Progression 2, 1, will give the Terms $\frac{1}{2}x^2 - xy$. Also the Terms, or rather Term, $-x\dot{y} + ay$, by expunging the points will become $-xy + ay$, which are only to be divided by Unity. So that leaving out the redundant Term $-xy$, we shall have the Fluential Equation $\frac{1}{2}x^2 - xy + ay = 0$. Now if we take the Fluxions of this Equation, we shall find by the foregoing Problem $x\dot{x} - \dot{x}y - x\dot{y} + ay = 0$, which being the same as the Equation given, we are to conclude our work is true. But if either of the Fluxional Equations $\dot{x}x - \dot{x}y + ay = 0$, or $\dot{x}x - x\dot{y} + ay = 0$ had been proposed, tho' by pursuing the foregoing method we should arrive at the Equation $\frac{1}{2}x^2 - xy + ay = 0$, for the relation of the Fluents; yet as this conclusion would not stand the test of this proof, we must reject it as erroneous, and have recourse to the following general Method; which will give the value of y in either of those Equations by an infinite Series, and therefore for use and practice will be the most commodious Solution.

5. As Velocities can be compared only with Velocities, and all other quantities with others of the same Species only; therefore in every Term of an Equation, the Fluxions must always ascend to the same number of Dimensions, that the homogeneity may not be destroyed. Whenever it happens otherwise, 'tis because some Fluxion, taken for Unity, is there understood, and therefore must be supply'd when occasion requires. The Equation $\dot{x}z + \dot{x}yx - az^2x^2 = 0$, by making $z = 1$, may become $\dot{x} + \dot{x}yx - ax^2 = 0$, and likewise *vice versâ*. And as this Equation virtually involves three variable quantities,

quantities, it will require another Equation, either Fluential or Fluxional, for a compleat determination, as has been already observed. So as the Equation $\dot{y}x = \dot{x}yy$, by putting $\dot{x} = 1$ becomes $\dot{y}x = yy$; in like manner this Equation requires and supposes the other.

6, 7, 8, 9, 10, 11. Here we are taught some useful Reductions, in order to prepare the Equation for Solution. As when the Equation contains only two flowing Quantities with their Fluxions, the ratio of the Fluxions may always be reduced to simple Algebraic Terms. The Antecedent of the Ratio, or its Fluent, will be the quantity to be extracted; and the Consequent, for the greater simplicity, may be made Unity. Thus the Equation $2\dot{x} + 2x\dot{x} - y\dot{x} - \dot{y} = 0$ is reduced to this, $\frac{\dot{y}}{\dot{x}} = 2 + 2x - y$, or making $\dot{x} = 1$, 'tis $\dot{y} = 2 + 2x - y$. So the Equation $\dot{y}a - \dot{y}x - \dot{x}a + \dot{x}x - \dot{x}y = 0$, making $\dot{x} = 1$, will become $\dot{y} = \left(\frac{a-x+y}{a-x} = 1 + \frac{y}{a-x} =\right) 1 + \frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}$, &c. by Division. But we may apply the particular Solution to this Example, by which we shall have $\frac{1}{2}x^2 - xy - ax + ay = 0$, and thence $y = \frac{ax - \frac{1}{2}x^2}{a-x}$. Thus the Equation $\dot{y}\dot{y} = \dot{x}\dot{y} + \dot{x}\dot{x}x$, making $\dot{x} = 1$, becomes $\dot{y}\dot{y} = \dot{y} + xx$, and extracting the square-root, 'tis $\dot{y} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + xx} = \frac{1}{2} \pm$ the Series $\frac{1}{2} + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. that is, either $\dot{y} = 1 + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. or $\dot{y} = -x^2 + x^4 - 2x^6 + 5x^8 - 14x^{10}$, &c. Again, the Equation $\dot{y}^3 + ax\dot{x}^2\dot{y} + a^2\dot{x}^2\dot{y} - x^3\dot{x}^3 - 2\dot{x}^3a^3 = 0$, putting $\dot{x} = 1$, becomes $\dot{y}^3 + ax\dot{y} + a^2\dot{y} - x^3 - 2a^3 = 0$. Now an affected Cubic Equation of this form has been resolved before, (pag. 12.) by which we shall have $\dot{y} = a - \frac{1}{4}x + \frac{xx}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c.

12. For the sake of perspicuity, and to fix the Imagination, our Author here introduces a distinction of Fluents and Fluxions into *Relate* and *Correlate*. The Correlate is that flowing Quantity which he supposes to flow equably, which is given, or may be assumed, at any point of time, as the known measure or standard, to which the Relate Quantity may be always compared. It may therefore very properly denote Time; and its Velocity or Fluxion, being an uniform and constant quantity, may be made the Fluxional Unit, or the known measure of the Fluxion (or of the rate of flowing) of the Relate Quantity. The Relate Quantity, (or Quantities if several

ral are concern'd,) is that which is suppos'd to flow inequably, with any degrees of acceleration or retardation; and its inequability may be measured, or reduced as it were to equability, by constantly comparing it with its corresponding Correlate or equable Quantity. This therefore is the Quantity to be found by the Problem, or whose Root is to be extracted from the given Equation. And it may be conceived as a Space described by the inequable Velocity of a Body or Point in motion, while the equable Quantity, or the Correlate, represents or measures the time of description. This may be illustrated by our common Mathematical Tables, of Logarithms, Sines, Tangents, Secants, &c. In the Table of Logarithms, for instance, the Numbers are the Correlate Quantity, as proceeding equably, or by equal differences, while their Logarithms, as a Relate Quantity, proceed inequably and by unequal differences. And this resemblance would more nearly obtain, if we should suppose infinite other Numbers and their Logarithms to be interpolated, (if that infinite Number be every where the same,) so as that in a manner they may become continuous. So the Arches or Angles may be consider'd as the Correlate Quantity, because they proceed by equal differences, while the Sines, Tangents, Secants, &c. are as so many Relate Quantities, whose rate of increase is exhibited by the Tables.

13, 14, 15, 16, 17. This Distribution of Equations into Orders, or Classes, according to the number of the flowing Quantities and their Fluxions, tho' it be not of absolute necessity for the Solution, may yet serve to make it more expedite and methodical, and may supply us with convenient places to rest at.

SECT. II. *Solution of the first Case of Equations.*

18, 19, 20, 21, 22, 23. **T**HE first Case of Equations is, when the Quantity $\frac{y}{x}$, or what supplies its place, can always be found in Terms composed of the Powers of x , and known Quantities or Numbers. These Terms are to be multiply'd by x , and to be divided by the Index of x in each Term, which will then exhibit the Value of y . Thus in the Equation $y^2 = xy + x^2x^2$, it has been found that $\frac{y}{x} = 1 + x^2 - x^4 + 2x^6 - 5x^8 + 14x^{10}$, &c. Therefore $\frac{yx}{x} = x + x^3 - x^5 + 2x^7 - 5x^9 + 14x^{11}$, &c. and consequently $y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 - \frac{5}{9}x^9 + \frac{14}{11}x^{11}$, &c. as may easily be proved by the direct Method.

But

But this, and the like Equations, may be resolved more readily by a Method form'd in imitation of some of the foregoing Analyses, after this manner. In the given Equation make $x = 1$; then it will be $y^2 = y + x^2$, which is thus resolved:

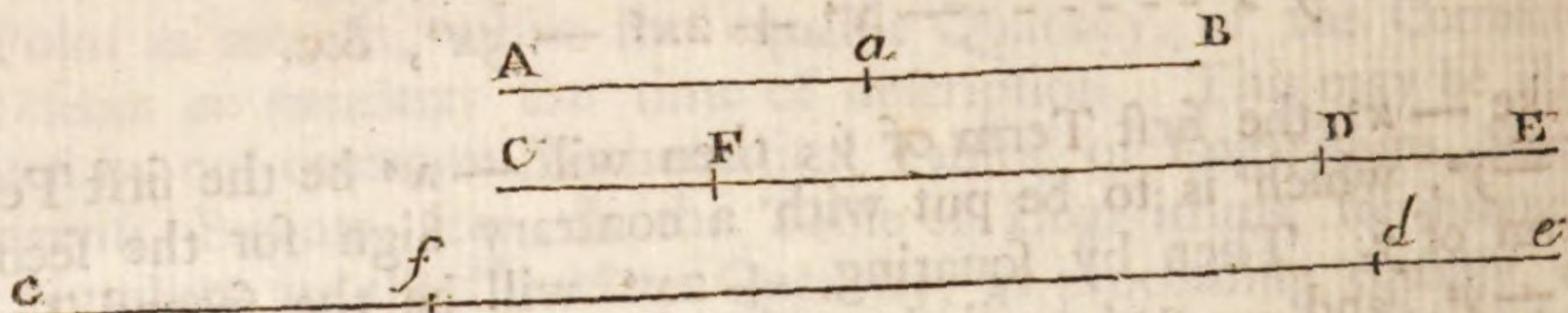
$$\begin{aligned} y \} &= -x^2 + x^4 - 2x^6 + 5x^8, \&c. \\ -y^2 \} &-----x^4 + 2x^6 - 5x^8, \&c. \end{aligned}$$

Make $-x^2$ the first Term of y ; then will $-x^4$ be the first Term of $-y^2$, which is to be put with a contrary Sign for the second Term of y . Then by squaring, $+2x^6$ will be the second Term of $-y^2$, and $-2x^6$ will be the third Term of y . Therefore $-5x^8$ will be the third Term of $-y^2$, and $+5x^8$ will be the fourth Term of y ; and so on. Therefore taking the Fluents, $y = -\frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{2}{7}x^7 + \frac{5}{9}x^9, \&c.$ which will be one Root of the Equation. And if we subtract this from x , we shall have $y = x + \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{2}{7}x^7 - \frac{5}{9}x^9, \&c.$ for the other Root.

So if $\frac{y}{x} = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2}, \&c.$ that is, if $\frac{yx}{x} = ax - \frac{1}{4}x^2 + \frac{x^3}{64a} + \frac{131x^4}{512a^2}, \&c.$ then $y = ax - \frac{1}{8}x^2 + \frac{x^3}{192a} + \frac{131x^4}{2048a^2}, \&c.$ If $\frac{y}{x} = \frac{1}{x^3} - \frac{1}{x^2} + \frac{a}{x^{\frac{1}{2}}} - x^{\frac{1}{2}} + x^{\frac{3}{2}}, \&c.$ or $\frac{yx}{x} = x^{-2} - x^{-1} + ax^{\frac{1}{2}} - x^{\frac{3}{2}} + x^{\frac{5}{2}}, \&c.$ then $y = -\frac{1}{2}x^{-2} + x^{-1} + 2ax^{\frac{1}{2}} - \frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}, \&c.$ If $\frac{y}{x} = \frac{2b^2c}{\sqrt{ay^3}} + \frac{3y^2}{a+b} + \sqrt{by+cy}$, or $\frac{xy}{y} = \frac{2b^2cy^{-\frac{1}{2}}}{a^{\frac{1}{2}}} + \frac{3y^3}{a+b} + y^{\frac{3}{2}}\sqrt{b+c}$, then $x = -\frac{4b^2c}{\sqrt{ay}} + \frac{y^3}{a+b} + \frac{2}{3}y^{\frac{3}{2}}\sqrt{b+c}$. If $\frac{y}{z} = z^{\frac{2}{3}}$, or $\frac{yz}{z} = z^{\frac{5}{3}}$, then $y = \frac{3}{5}z^{\frac{5}{3}}$. If $\frac{y}{x} = \frac{ab}{cx^{\frac{1}{3}}} = \frac{ab}{c}x^{-\frac{1}{3}}$, or $\frac{yx}{x} = \frac{ab}{c}x^{\frac{2}{3}}$, then $y = \frac{3ab}{2c}x^{\frac{2}{3}}$.

Lastly, if $\frac{y}{x} = \frac{a}{x}$, or $\frac{yx}{x} = a = ax^0$; dividing by the Index 0, it will be $y = \frac{a}{0}$, or y is infinite. That this Expression, or value of y , must be infinite, is very plain. For as 0 is a vanishing quantity, or less than any assignable quantity, its Reciprocal $\frac{1}{0}$ or $\frac{a}{0}$ must be bigger than any assignable quantity, that is, infinite.

Now that this quantity ought to be infinite, may be thus proved. In the Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, let AB represent the constant quantity a , and in CE let a point move equably from C towards E, and describe the Line CDE, of which let any indefinite part CD be x , and its equable Velocity in D, (and every where else,) is represented



by $\dot{x}$. Also let a point move from a distant point c along the Line cde , with an inequable Velocity, and let the Line described in the same time, or the indefinite part of it cd , be call'd y , and let the Velocity in d be call'd $\dot{y}$. The Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$ must always obtain, whatever the contemporaneous values of x and y may be; or in the whole Motion the constant Line AB (a) must be to the variable Line CD (x), as the Velocity in d ($\dot{y}$) is to the Velocity in D ($\dot{x}$). But at the beginning of the Motion, or when CD (x) was infinitely little, as the ratio of AB to CD was then greater than any assignable ratio, so also was the ratio $\frac{\dot{y}}{\dot{x}}$ of the Velocities, or the Velocity $\dot{y}$ was infinitely greater than the Velocity $\dot{x}$. But an infinite Velocity must describe an infinite Space in a finite time, or the point c is at an infinite distance from the point d , that is, y is an infinite quantity.

24, 25. But to avoid such infinite Expressions, from whence we can conclude nothing; we are at liberty to change the initial points of the Fluents, by which their Rate of flowing, (the only thing to be here regarded,) will not at all be affected. Thus in the foregoing Figure, we supposed the points D and d to be such, as limited the contemporaneous Fluents, or in which the two describing points were found at the same time. Let F and f be any other two such points, and then the finite Line CF $= b$ will be contemporaneous to, or will correspond with, the infinite Line $cf = c$; and FD, which may be made the new x , will correspond to fd , which will be the new y . So that in the given Equation $\frac{\dot{y}}{\dot{x}} = \frac{a}{x}$, instead of

x we may write $b + x$, and we shall have $\frac{y}{x} = \frac{a}{b+x}$, and then by Multiplication and Division it is $\frac{yx}{x} = \left(\frac{ax}{b+x} = \right) \frac{ax}{b} - \frac{ax^2}{b^2} + \frac{ax^3}{b^3} - \frac{ax^4}{b^4}$, &c. and therefore $y = \frac{ax}{b} - \frac{ax^2}{b^2} + \frac{ax^3}{b^3} - \frac{ax^4}{b^4}$, &c.

26. So if $\frac{y}{x} = \frac{2}{1+x} + 2 - 2x - xx$, because of the Term $\frac{2}{x}$, which would give an infinite value for y , we may write $1+x$ instead of x , and we shall then have $\frac{y}{x} = \frac{2}{1+x} + 2 - 2x - xx$, or $\frac{yx}{x} = \frac{2x}{1+x} + 2x - 2x^2 - x^3$, or by Division $\frac{yx}{x} = 4x - 4x^2 + x^3 - 2x^4 + 2x^5$, &c. and therefore $y = 4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5$, &c.

Or the Equation $\frac{y}{x} = \frac{2}{1+x} + 2 - 2x - x^2$, that is $y + xy = 4 - 3x^2 - x^3$, may be thus resolved:

$$\begin{aligned} y \} &= 4 - 3x^2 - x^3 \\ + xy \} &= -4x + 4x^2 - x^3 + 2x^4, \text{ \&c.} \\ \hline y &= 4 - 4x + x^2 - 2x^3 + 2x^4, \text{ \&c.} \\ y &= 4x - 2x^2 + \frac{1}{3}x^3 - \frac{1}{2}x^4 + \frac{2}{5}x^5, \text{ \&c.} \end{aligned}$$

Make 4 the first Term of y , then $4x$ will be the first Term of xy , and consequently $-4x$ will be the second Term of y . Then $-4x^2$ will be the second Term of xy , and therefore $+4x^2 - 3x^2$, or x^2 , will be the third Term of y ; and so on.

27. So if $\frac{y}{x} = x^{-\frac{1}{2}} + x^{-1} - x^{\frac{1}{2}}$, because of the Term x^{-1} change x into $1-x$, then $\frac{y}{x} = \frac{1}{\sqrt{1-x}} + \frac{1}{1-x} - \sqrt{1-x}$. But

$$\begin{aligned} \text{by the foregoing Methods of Reduction 'tis } \frac{1}{1-x} &= 1 + x + x^2 + x^3, \text{ \&c. and } \sqrt{1-x} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3, \text{ \&c. and} \\ \frac{1}{\sqrt{1-x}} &= \frac{1}{1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3, \text{ \&c.}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3, \text{ \&c.} \end{aligned}$$

Therefore collecting these according to their Signs, 'tis $\frac{y}{x} = 1 + 2x + \frac{3}{2}x^2 + \frac{27}{16}x^3$, &c. that is $\frac{yx}{x} = x + 2x^2 + \frac{3}{2}x^3 + \frac{27}{16}x^4$, &c. and therefore $y = x + x^2 + \frac{1}{2}x^3 + \frac{27}{16}x^4$, &c.

28. So if the given Equation were $\frac{y}{x} = \frac{c^2x}{c^3 - 3c^2x + 3cx^2 - x^3} = \frac{c^2x}{(c-x)^3}$; change the beginning of x , that is, instead of x write $c-x$,

$c = x$, then $\frac{\dot{y}}{x} = \frac{c^3 - c^2x}{x^3} = c^3x^{-3} - c^2x^{-2}$, or $\frac{\dot{y}x}{x} = c^3x^{-2} - c^2x^{-1}$, and therefore $y = -\frac{1}{2}c^3x^{-2} + c^2x^{-1}$.

SECT. III. Solution of the second Case of Equations.

29, 30. **E**quations belonging to this second case are those, wherein the two Fluents and their Fluxions, suppose x and y , $\dot{x}$ and $\dot{y}$, or any Powers of them, are promiscuously involved. As our Author's Analyses are very intelligible, and seem to want but little explication, I shall endeavour to resolve his Examples in something an easier and simpler manner, than is done here; by applying to them his own artifice of the Parallelogram, when needful, or the properties of a combined Arithmetical Progression *in plano*, as explain'd before: As also the Methods before made use of, in the Solution of affected Equations.

31. The Equation $yax - xxy - aax = 0$ by a due Reduction becomes $\frac{\dot{y}}{x} = \frac{y}{a} + \frac{a}{x}$, in which, because of the Term $\frac{a}{x}$ there is occasion for a Transmutation, or to change the beginning of the Correlate Quantity x . Assuming therefore the constant quantity b , we may put $\frac{\dot{y}}{x} = \frac{y}{a} + \frac{a}{b+x}$, whence by Division will be had $\frac{\dot{y}}{x} = \frac{y}{a} + \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4}$, &c. which Equation is then prepared for the Author's Method of Solution.

But without this previous Reduction to an infinite Series, and the Resolution of an infinite Equation consequent thereon, we may perform the Solution thus, in a general manner. The given Equation is now $\frac{\dot{y}}{x} = \frac{y}{a} + \frac{a}{b+x}$, or putting $\dot{x} = 1$, it is $aby + axy = by + yx + a^2$, which may be thus resolved:

$$\begin{aligned} \left. \begin{aligned} aby &= a^2 + \frac{ab-a^2}{b}x + \frac{2a^2+b^2-ab}{2b^2}x^2 + \frac{b^3+2a^2b-ab^2-6a^3}{6ab^3}x^3, \text{ \&c.} \\ +axy &= \frac{a^2}{b}x + \frac{ab-aa}{bb}x^2 + \frac{2a^2+b^2-ab}{2b^3}x^3, \text{ \&c.} \\ -by &= \frac{a-b}{2b}x^2 + \frac{ab-2a^2-b^2}{6ab^2}x^3, \text{ \&c.} \\ -xy &= \frac{a}{b}x^2 + \frac{a-b}{2b^2}x^3, \text{ \&c.} \end{aligned} \right\} \\ \dot{y} &= \frac{a}{b} + \frac{b-a}{b^2}x + \frac{2a^2+b^2-ab}{2ab^3}x^2 + \frac{b^3+2a^2b-ab^2-6a^3}{6a^2b^4}x^3, \text{ \&c.} \\ y &= \frac{a}{b}x + \frac{b-a}{2b^2}x^2 + \frac{2a^2+b^2-ab}{6ab^3}x^3 + \frac{b^3+2a^2b-ab^2-6a^3}{24a^2b^4}x^4, \text{ \&c.} \end{aligned}$$

Disposing

Disposing the Terms as you see is done here, make a^2 the first Term of aby , then $\frac{a}{b}$ will be the first Term of y , and thence $\frac{a}{b}x$ will be the first Term of y . So that $\frac{a^2}{b}x$ will be the first Term of axy , and $-ax$ will be the first Term of $-by$. These two together, or $\frac{a^2}{b}x - ax = \frac{a^2 - ab}{b}x$, with a contrary Sign, must be put down for the second Term of aby . Therefore the second Term of y will be $\frac{b-a}{b^2}x$, and the like Term of y will be $\frac{b-a}{2b^2}x^2$. Then the second Term of axy will be $\frac{ab-a^2}{b^2}x^2$, and the second Term of $-by$ will be $\frac{a-b}{2b}x^2$, and the first Term of $-xy$ will be $-\frac{a}{b}x^2$. These three together make $\frac{ab-2a^2-b^2}{2b^2}x^2$, which with a contrary Sign must be made the third Term of aby . Therefore the third Term of y will be $\frac{2a^2+b^2-ab}{2ab^3}x^2$, and the third Term of y will be $\frac{2a^2+b^2-ab}{6ab^3}x^3$. And so on. Here in a particular case if we make $b=a$, we shall have the simple Series $y = x + \frac{x^3}{3a^2} - \frac{x^4}{6a^3}$, &c.

Or if we would have a descending Series for the Root y of this Equation, we may proceed as follows:

$$\begin{array}{rcl}
 -xy &] & = a^2 - \overline{a+b} \times a^2x^{-1} + \overline{2a^2+2ab+b^2} \times a^2x^{-2}, \&c. \\
 -by &] & - - - + \quad \quad \quad a^2bx^{-1} - \quad \quad \quad \overline{a+b} \times a^2bx^{-2}, \&c. \\
 +axy &] & - - - + \quad \quad \quad a^3x^{-1} - \quad \quad \quad \overline{a+b} \times 2a^3x^{-2}, \&c. \\
 +aby &] & - - - - - - - - - + \quad \quad \quad a^3bx^{-2}, \&c. \\
 y & = & -a^2x^{-1} + \overline{a+b} \times a^2x^{-2} - \overline{2a^2+2ab+b^2} \times a^2x^{-3}, \&c. \\
 y & = & a^2x^{-2} - \overline{a+b} \times 2a^2x^{-3}, \&c.
 \end{array}$$

Dispose the Terms as you see, and make a^2 the first Term of the Series $-xy$; then will $-\frac{a^2}{x}$ be the first Term of y , and a^2x^{-2} will be the first Term of y . Then will $+a^2bx^{-1}$ be the first Term of $-by$, and a^3x^{-1} will be the first Term of axy , which together make $\overline{a+b} \times a^2x^{-1}$; this therefore with a contrary Sign must be the second Term of $-xy$. Then the second Term of y will be $\overline{a+b} \times a^2x^{-2}$, and the second Term of y will be $-\overline{a+b} \times 2a^2x^{-3}$. Therefore the second Term of $-by$ will be $-\overline{a+b} \times a^2bx^{-2}$, and

and the second Term of axy will be $-\overline{a+b} \times 2a^3x^{-2}$, and the first Term of aby will be a^3bx^{-2} ; which three together make $-\overline{2a^2 + 2ab + b^2} \times a^2x^{-2}$. This with a contrary Sign must be the third Term of $-xy$, which will give $-\overline{2a^2 + 2ab + b^2} \times a^2x^{-3}$ for the third Term of y ; and so on. Here if we make $b = a$, then $y = -\frac{a^2}{x} + \frac{2a^3}{x^2} - \frac{5a^4}{x^3}$, &c.

And these are all the Series, by which the value of y can be exhibited in this Equation, as may be proved by the Parallelogram. For that Method may be extended to these Fluxional Equations, as well as to Algebraical or Fluential Equations. To reduce these Equations within the Limits of that Rule, we are to consider, that as Ax^m may represent the initial Term of the Root y , in both these kinds of Equations, or because it may be $y = Ax^m$, &c. so in Fluxional Equations (making $x = 1$, we shall have also $\dot{y} = mAx^{m-1}$, &c. or writing y for Ax^m , &c. 'tis $\dot{y} = myx^{-1}$, &c. So that in every Term of the given Equation, in which $\dot{y}$ occurs, or the Fluxion of the Relate Quantity, we may conceive it to take away one Dimension from the Correlate Quantity, suppose x , and to add it to the Relate Quantity, suppose y ; according to which Reduction we may insert the Terms in the Parallelogram. And we are to make a like Reduction for all the Powers of the Fluxion of the Relate Quantity. This will bring all Fluxional Equations to the Case of Algebraic Equations, the Resolution of which has been so amply treated of before.

Thus in the present Equation $aby + ax\dot{y} = by + yx + aa$, the Terms must be inserted in the Parallelogram, as if yx^{-1} were substituted instead of $\dot{y}$; so that the Indices will stand as in the Margin, and the Ruler will give only two Cases of external Terms. Or rather, if we would reduce this Equation to the form of a double Arithmetical Scale, as explain'd before, we should have it in this form. Here in the first Column are contain'd those Terms which have y of one Dimension, or what is equivalent to it. In the second Column is $-a^2$, or y of no Dimensions. Also in the first Line is $-xy$, or such Terms in which x is of one Dimension. In the second Line are the Terms $-\frac{by}{+axy} \} -a^2$, which have no Dimensions of x , because $+axy$ is regarded as if it were ay . Lastly, in the third line is aby , or the Term in which x is of one negative Dimension,

	$m+1$
o	m
	$m-1$

$$\left. \begin{array}{l} -xy \\ -by \\ +axy \\ +aby \end{array} \right\} -a^2 = 0.$$

Dimension, because $+aby$ is consider'd as if it were $+abx^{-1}y$. And these Terms being thus dispos'd, it is plain there can be but two Cases of external Terms, which we have already discuss'd.

32. If the proposed Equation be $\frac{y}{x} = 3y - 2x + \frac{x}{y} - \frac{2y}{xx}$, or making $x = 1$, 'tis $-y + 3y - 2x + xy^{-1} - 2yx^{-2} = 0$; the Solution of which we shall attempt without any preparation, or without any new interpretation of the Quantities. First, the Terms are to be dispos'd according to a double Arithmetical Scale, the Roots of which are y and x , and then they will stand as in the Margin. The Method of doing this with certainty in all cases is as follows. I observe in the Equation there are three powers of y , which are y^1 , y^0 , and y^{-1} ; therefore I place these in order at the top of the Table. I observe likewise that there are four Powers of x , which are x^1 , x^0 , x^{-1} , and x^{-2} , which I place in order in a Column at the right hand; or it will be enough to conceive this to be done. Then I insert every Term of the Equation in its proper place, according to its Dimensions of y and x in that Term; filling up the vacancies with Asterisks, to denote the absence of the Terms belonging to them. The Term $-y$ I insert as if it were $-yx^{-1}$, as is explain'd before. Then we may perceive, that if we apply the Ruler to the exterior Terms, we shall have three cases that may produce Series; for the fourth case, which is that of direct ascent or descent, is always to be omitted, as never affording any Series. To begin with the descending Series, which will arise from the two external Terms $-2x$ and $+xy^{-1}$. The Terms are to be dispos'd, and the Analysis to be perform'd, as here follows:

$$\begin{array}{c|ccc} & y^1 & y^0 & y^{-1} \\ \hline x^1 & * & -2x & +xy^{-1} \\ x^0 & +3y & * & * \\ x^{-1} & -y & * & * \\ x^{-2} & -2yx^{-2} & * & * \end{array} \Bigg\} = 0.$$

$$\begin{array}{l} xy^{-1} \left\{ \begin{array}{l} = 2x - \frac{3}{2} - \frac{9}{8}x^{-1} - \frac{17}{16}x^{-2}, \&c. \\ - - - + \frac{3}{2} + \frac{9}{8}x^{-1} + \frac{17}{16}x^{-2}, \&c. \\ -y \quad \quad \quad * + \frac{3}{8}x^{-2}, \&c. \\ -2yx^{-2} \quad \quad \quad - x^{-2}, \&c. \end{array} \right. \\ y = \frac{1}{2} + \frac{3}{8}x^{-1} + \frac{9}{16}x^{-2} + \frac{11}{128}x^{-3}, \&c. \end{array}$$

Make $xy^{-1} = 2x$, &c. then $y^{-1} = 2$, &c. and by Division $y = \frac{1}{2}$, &c. Therefore $3y = \frac{3}{2}$, &c. and consequently $xy^{-1} = *$ $-\frac{3}{2}$, &c. or $y^{-1} = *$ $-\frac{3}{2}x^{-1}$, &c. and by Division $y = *$ $+\frac{9}{8}x^{-1}$, &c. Therefore $3y = *$ $+\frac{9}{8}x^{-1}$, &c. and consequently $xy^{-1} = **$ $-\frac{9}{8}x^{-1}$, &c. So that $y^{-1} = **$ $-\frac{9}{8}x^{-2}$, &c. and by Division $y = **$ $+\frac{17}{16}x^{-2}$, &c. Then $3y = **$ $+\frac{17}{16}x^{-2}$, &c. and

$-y = * + \frac{3}{8}x^{-2}$, &c. and $-2yx^{-2} = -x^{-2}$, &c. These three together make $+\frac{1}{8}x^{-2}$, and therefore $xy^{-1} = *** - \frac{1}{8}x^{-2}$, &c. so that $y = *** + \frac{1}{8}x^{-2}$, &c. And so on.

Another descending Series will arise from the two external Terms $+3y$ and $-2x$, which may be thus extracted:

$$\left. \begin{array}{l} 3y \\ +xy^{-1} \\ -y \\ -2yx^{-2} \end{array} \right\} \begin{array}{l} = 2x - \frac{5}{8} + \frac{1}{2}\frac{7}{4}x^{-1} - \frac{1}{4}\frac{5}{8}x^{-2}, \text{ \&c.} \\ - \quad - \quad + \frac{3}{2} + \frac{5}{8}x^{-1} - \frac{1}{4}\frac{3}{8}x^{-2}, \text{ \&c.} \\ - \quad - \quad - \frac{2}{3} \quad * \quad + \frac{7}{2}x^{-2}, \text{ \&c.} \\ - \quad - \quad - \quad - \frac{4}{3}x^{-1} + \frac{5}{9}x^{-2}, \text{ \&c.} \end{array}$$

$$y = \frac{2}{3}x - \frac{5}{8} + \frac{1}{2}\frac{7}{4}x^{-1} - \frac{1}{4}\frac{5}{8}x^{-2}, \text{ \&c.}$$

$$y^{-1} = \frac{3}{2}x^{-1} + \frac{5}{8}x^{-2} - \frac{1}{4}\frac{3}{8}x^{-3}, \text{ \&c.}$$

$$\dot{y} = \frac{2}{3} * - \frac{1}{2}\frac{7}{4}x^{-2}, \text{ \&c.}$$

Make $3y = 2x$, &c. then $y = \frac{2}{3}x$, &c. and (by Division) $y^{-1} = \frac{3}{2}x^{-1}$, &c. and $xy^{-1} = \frac{3}{2}$, &c. and $-y = -\frac{2}{3}$, &c. Therefore $3y = * - \frac{5}{8}$, &c. and $y = * - \frac{5}{8}$, &c. and (by Division) $xy^{-1} = * \frac{5}{8}x^{-1}$, &c. and $-y = * 0$, &c. and $-2yx^{-2} = -\frac{4}{3}x^{-1}$, &c. Therefore $3y = *** + \frac{1}{2}\frac{7}{4}x^{-1}$, &c. and $y = *** + \frac{1}{8}x^{-2}$, &c. &c.

The ascending Series in this Equation will arise from the two external Terms $-2yx^{-2}$ and xy^{-1} ; or multiplying the whole Equation by $-y$, (that one of the external Terms may be clear'd from y .) we shall have $yy - 3y^2 + 2xy - x + 2y^2x^{-2} = 0$, of which the Resolution is thus:

$$\left. \begin{array}{l} 2y^2x^{-2} \\ + yy \\ + 2xy \\ - 3y^2 \end{array} \right\} \begin{array}{l} = x^2 * - \frac{3}{4}x^{\frac{4}{2}} - \frac{2}{\sqrt{2}}x^{\frac{5}{2}} + \frac{9}{4}x^3, \text{ \&c.} \\ - \quad - \quad + \frac{3}{4}x^2 \quad * \quad - \frac{3}{4}x^3, \text{ \&c.} \\ - \quad - \quad - \quad - \quad + \frac{2}{\sqrt{2}}x^{\frac{5}{2}} \quad *, \text{ \&c.} \\ - \quad - \quad - \quad - \quad - \quad - \frac{3}{2}x^3, \text{ \&c.} \end{array}$$

$$y = \frac{1}{\sqrt{2}}x^{\frac{1}{2}} * - \frac{3}{8\sqrt{2}}x^{\frac{5}{2}} - \frac{1}{2}x^3 + \frac{135}{128\sqrt{2}}x^{\frac{7}{2}}, \text{ \&c.}$$

$$\dot{y} = \frac{3}{2\sqrt{2}}x^{\frac{1}{2}} * - \frac{15}{16\sqrt{2}}x^{\frac{3}{2}} - \frac{3}{2}x^2, \text{ \&c.}$$

Make $2y^2x^{-2} = x$, &c. then $y^2 = \frac{1}{2}x^3$, &c. and $y = \frac{1}{\sqrt{2}}x^{\frac{3}{2}}$, &c. Here because of the fractional Indices, and that the first Term of $+2xy$, or $+\frac{2}{\sqrt{2}}x^{\frac{5}{2}}$, may be afterwards admitted, we must take 0 for the second Term of $2y^2x^{-2}$, and therefore for the second Term of

of y . Then $\dot{y}y = \frac{1}{4}x^2$, &c. and consequently $2y^2x^{-2} = \dots - \frac{1}{4}x^2$, &c. and $y^2 = \dots - \frac{1}{8}x^4$, &c. and by extracting the square-root, $y = \dots - \frac{1}{8\sqrt{2}}x^2$, &c. Then $\dot{y}y = \dots + 0$, &c. and $2xy = \frac{2}{\sqrt{2}}x^{\frac{3}{2}}$, &c. and therefore $2y^2x^{-2} = \dots - \frac{2}{\sqrt{2}}x^{\frac{5}{2}}$, &c. and $y = \dots - \frac{1}{2}x^3$, &c. &c.

33, 34. The Author's Process of Resolution, in this and the following Examples, is very natural, simple, and intelligible; it proceeds *seriatim* & *terminatim*, by passing from Series to Series, and by gathering Term after Term, in a kind of circulating manner, of which Method we have had frequent instances before. By this means he collects into a Series what he calls the Sum, which Sum is the value of $\frac{\dot{y}}{x}$ or of the Ratio of the Fluxions of the Relate and Correlate in the given Equation; and then by the former Problem he obtains the value of y . When I first observed this Method of Solution, in this Treatise of our Author's, I confess I was not a little pleased; it being nearly the same, and differing only in a few circumstances that are not material, from the Method I had happen'd to fall into several years before, for the Solution of Algebraical and Fluxional Equations. This Method I have generally pursued in the course of this work, and shall continue to explain it farther by the following Examples.

The Equation of this Example $1 - 3x + y + x^2 + xy - \dot{y} = 0$ being reduced to the form of a double Arithmetical Scale, will stand as here in the Margin; and the Ruler will discover two cases to be try'd, of which one may give us an ascending, and the other a descending Series for the Root y . And first for the ascending Series.

$$\begin{array}{r|l} & \begin{array}{cc} y^1 & y^0 \end{array} \\ \hline x^2 & \begin{array}{cc} * & + x^2 \end{array} \\ x^1 & \begin{array}{cc} + xy & - 3x \end{array} \\ x^0 & \begin{array}{cc} + y & + 1 \end{array} \\ x^{-1} & \begin{array}{cc} - \dot{y} & * \end{array} \end{array} \Bigg\} = 0.$$

$$\begin{array}{l} \dot{y} \Bigg\} = 1 - 3x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5, \text{ \&c.} \\ -y \Bigg\} \quad \quad \quad + x \quad \quad \quad * \\ -xy \Bigg\} \quad \quad \quad - x + x^2 - \frac{1}{3}x^3 + \frac{1}{6}x^4 - \frac{1}{30}x^5, \text{ \&c.} \\ \quad \quad \quad - x^2 + x^3 - \frac{1}{3}x^4 + \frac{1}{6}x^5, \text{ \&c.} \\ \dot{y} = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5, \text{ \&c.} \\ y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6, \text{ \&c.} \end{array}$$

The Terms being disposed as you see, make $\dot{y} = 1$, &c. then $y = x$, &c. Therefore $-y = -x$, &c. the Sign of which Term being changed, it will be $\dot{y} = \dots + x - 3x$, &c. $= \dots - 2x$, &c.

Now in this Example, because the Series $\frac{y}{a} + \frac{xy}{a^2} + \frac{x^2y}{a^3} + \frac{x^3y}{a^4}, \&c.$ is equal to $\frac{y}{a-x}$, it will be $y = \frac{y}{a-x} + 1$, or $ay - xy = y + a - x$, that is, $yx + ax - xx - ay + xy = 0$; which Equation, by the particular Solution before deliver'd, will give the relation of the Fluents $yx - ay + ax - \frac{1}{2}x^2 = 0$. Hence $y = \frac{ax - \frac{1}{2}xx}{a-x}$, and by Division $y = x + \frac{x^2}{2a} + \frac{x^3}{2a^2} + \frac{x^4}{2a^3}, \&c.$ as found above.

37. The Equation of this Example being tabulated, or reduced to a double Arithmetical Scale, will stand as here in the Margin. Where it may be observed, that because of the Series proceeding both ways *ad infinitum*, there can be but one case of exterior Terms, of which the Solution here follows:

	y^0	y^1	y^2	y^3	$y^4, \&c.$
x^{-1}	*	$-y$	*	*	*
x^0	*	*	$+y^2$	$+y^3$	$+y^4, \&c.$
x^1	$-3x$	$+3xy$	$-xy^2$	$-xy^3$	$-xy^4, \&c.$
x^2	$-6x^2$	$+6x^2y$	*	*	*
x^3	$-8x^3$	$+8x^3y$	*	*	*
x^4	$-10x^4$	$+10x^4y$	*	*	*
x^5	$-12x^5$	$+12x^5y$	*	*	*
x^6	$-14x^6$	$\&c.$			
$\&c.$	$\&c.$				

$$\begin{array}{l}
 y \\
 - 3xy \\
 - 6x^2y \\
 - y^2 \\
 - 8x^3y \\
 + xy^2 \\
 - 10x^4y \\
 - y^3 \\
 \&c.
 \end{array}
 \left\{
 \begin{array}{l}
 = -3x - 6x^2 - 8x^3 - 10x^4 - 12x^5 - 14x^6, \&c. \\
 - \frac{3}{2}x^3 - 12\frac{3}{4}x^4 - 29\frac{5}{8}x^5 - 59\frac{2}{5}x^6, \&c. \\
 + \frac{3}{2}x^3 + 6x^4 + \frac{7}{8}x^5 + \frac{27}{10}x^6, \&c. \\
 + 9x^4 + 12x^5 + \frac{7}{4}x^6, \&c. \\
 - \frac{3}{4}x^4 - 6x^5 - \frac{107}{8}x^6, \&c. \\
 + 12x^5 + 16x^6, \&c. \\
 + \frac{3}{4}x^5 + 6x^6, \&c. \\
 + 15x^6, \&c. \\
 + \frac{27}{8}x^6, \&c.
 \end{array}
 \right. = 0.$$

$$y = -\frac{3}{2}x^2 - 2x^3 - \frac{25}{8}x^4 - \frac{21}{10}x^5 - \frac{111}{16}x^6 - \frac{367}{35}x^7, \&c.$$

Make $y = -3x, \&c.$ then $y = -\frac{3}{2}x^2, \&c.$ Then $y = * - 6x^2, \&c.$ and $y = * - 2x^3, \&c.$ Then $-3xy = +\frac{3}{2}x^3, \&c.$ and therefore $y = * * - \frac{3}{2}x^3 - 8x^3, \&c. = * * - \frac{25}{2}x^3, \&c.$ and $y = * * - \frac{25}{8}x^4, \&c.$ And so of the rest.

The Author here takes notice, that as the value of y is negative, and therefore contrary to that of x , it shews that as x increases, y must decrease, and on the contrary. For a negative Velocity is a Velocity backwarks, or whose direction is contrary to that which was

was suppos'd to be an affirmative Velocity. This Remark must take place hereafter, as often as there is occasion for it.

38. In this Example the Author puts x to represent the Relate Quantity, or the Root to be extracted, and y to represent the Correlate. But to prevent the confusion of Ideas, we shall here change x into y , and y into x , so that y shall denote the Relate, and x the Correlate Quantity, as usual. Let the given Equation therefore be $\frac{y}{x} = \frac{1}{2}x - 4x^2 + 2xy^{\frac{1}{2}} - \frac{4}{3}y^2 + 7x^{\frac{5}{2}} + 2x^3$, whose Root y is to be extracted. These Terms being disposed in a Table, will stand thus: And the Resolution will be as follows, taking $-y$ and $+\frac{1}{2}x$ for the two external Terms.

	y^2	$y^{\frac{3}{2}}$	y	$y^{\frac{1}{2}}$	1^0	
x^3	*	*	*	*	$+2x^3$	$\left. \begin{aligned} y &= \frac{1}{2}x - 4x^2 + 7x^{\frac{5}{2}} + 2x^3 \\ &+ x^2 - 2x^3 + 4x^{\frac{7}{2}} - \frac{4}{3}x^4, \&c. \\ - 2xy^{\frac{1}{2}} &= -x^2 + 2x^3 - 4x^{\frac{7}{2}} + 2x^4, \&c. \\ + \frac{4}{3}y^2 &= \dots\dots\dots + \frac{1}{3}x^4, \&c. \end{aligned} \right\}$
$x^{\frac{5}{2}}$	*	*	*	*	$+7x^{\frac{5}{2}}$	
x^2	*	*	*	*	$-4x^2$	
$x^{\frac{3}{2}}$	*	*	*	*	*	
x^1	*	*	*	*	$+2xy^{\frac{1}{2}} + \frac{1}{2}x$	
$x^{\frac{1}{2}}$	*	*	*	*	*	$y = \frac{1}{4}x^2 - x^3 + 2x^{\frac{7}{2}} + \frac{8}{9}x^{\frac{9}{2}} - \frac{41}{108}x^5, \&c.$
x^0	$-\frac{4}{3}y^2$	*	*	*	*	
$x^{-\frac{1}{2}}$	*	*	*	*	*	
x^{-1}	*	*	*	*	*	
$x^{-\frac{3}{2}}$	*	*	$-y$	*	*	

$\left. \begin{aligned} &+ 2xy^{\frac{1}{2}} + \frac{1}{2}x \\ &+ 2xy^{\frac{1}{2}} + \frac{1}{2}x \end{aligned} \right\} = 0.$

Make $y = \frac{1}{2}x$, &c. then $y = \frac{1}{4}x^2$, &c. Now because it is $y = * 0$, &c. it will be also $y = * 0$, &c. And whereas it is $y^{\frac{1}{2}} = \frac{1}{2}x$, &c. it will be $-2xy^{\frac{1}{2}} = -x^2$, &c. and therefore $y = * * + x^2 - 4x^2$, &c. $= * * - 3x^2$, &c. then $y = * * - x^3$, &c. Now because it is $y = * + 0$, &c. it will be also $y^{\frac{1}{2}} = * + 0$, &c. and $-2xy^{\frac{1}{2}} = * + 0$, &c. and consequently $y = * * * + 7x^{\frac{5}{2}}$, &c. and therefore $y = * * * + 2x^{\frac{7}{2}}$, &c. And so on.

There are two other cases of external Terms, which will supply us with two other Series for the Root y , but they will run too much into Surds. This may be sufficient to shew the universality of the Method, and how we are to proceed in like cases.

39. The Author shews here, that the same Fluxional Equation may often afford a great variety of Series for the Root, according as we shall introduce any constant quantity at pleasure. Thus the Equation of Art. 34. or $y = 1 - 3x + y + x^2 + xy$, may be resolved after the following general manner:

$$\begin{aligned} y &= 1 - 3x + x^2 \\ &+ a + x + 2ax^2 + \frac{5}{3}ax^3, \&c. \\ -y &= -a - x + x^2 - \frac{5}{3}ax^3, \&c. \\ -xy &= -ax - ax^2 - \frac{2}{3}ax^3, \&c. \\ &= -ax^2 + x^3 \end{aligned}$$

$$y = a + x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4, \&c. \\ + ax + ax^2 + \frac{2}{3}ax^3 + \frac{5}{12}ax^4$$

Here instead of making $y = 1$, &c. we may make $y = 0$, &c. and therefore $y = a$, &c. because then $y = 0$, &c. then $-y = -a$, &c. and consequently $y = * + a + 1$, &c. and therefore $y = * + ax + x$, &c. Then $-y = * - ax - x$, &c. and $-xy = -ax$, &c. and therefore $y = * + 2ax + x - 3x$, &c. $= * + 2ax - 2x$, &c. and then $y = * + ax^2 - x^2$, &c. Therefore $-y = * - ax^2 + x^2$, &c. and $-xy = * - ax^2 - x^2$, &c. and consequently $y = * + 2ax^2 + x^2$, &c. and $y = * + \frac{2}{3}ax^3 + \frac{1}{3}x^3$, &c. &c. Here if we make $a = 0$, we shall have the same value of y as was extracted before. And by whatever Number a is interpreted, so many different Series we shall obtain for y .

40. The Author here enumerates three cases, when an arbitrary Number should be assumed, if it can be done, for the first Term of the Root. First, when in the given Equation the Root is affected with a Fractional Dimension, or when some Root of it is to be extracted; for then it is convenient to have Unity for the first Term, or some other Number whose Root may be extracted without a Surd, if such Number does not offer itself of its own accord. As in the fourth Example 'tis $x = \frac{1}{4}y^2$, &c. and therefore we may easily have $x^{\frac{1}{2}} = \frac{1}{2}y$, &c. Secondly, it must be done, when by reason of the square-root of a negative Quantity, we should otherwise fall upon impossible Numbers. Lastly, we must assume such a Number, when otherwise there would be no initial Quantity, from whence to begin the computation of the Root; that is, when the Relate Quantity, or its Fluxion, affects all the Terms of the Equation.

41, 42, 43. The Author's Compendiums of Extraction are very curious, and shew the universality of his Method. As his several Processes want no explanation, I shall proceed to resolve his Examples by the foregoing general Method. As if the given Equation were $y = \frac{1}{y} - x^2$, or $y - y^{-1} = -x^2$, the Resolution might be thus:

y

$$\begin{aligned}
 \dot{y} & \left. \begin{aligned} &= 0 \\ &+ a^{-1} - a^{-3}x + \frac{3}{2}a^{-5}x^2 + \frac{1}{3}a^{-7}x^3, \&c. \\ &- a^{-1} + a^{-3}x - \frac{3}{2}a^{-5}x^2 + \frac{5}{2}a^{-7}x^3, \&c. \\ &- \frac{1}{3}a^{-7}x^3 \end{aligned} \right\} \\
 y &= a + \frac{x}{a} - \frac{x^2}{2a^3} + \frac{x^3}{2a^5} - \frac{5x^4}{8a^7}, \&c. \\
 &\quad - \frac{1}{3}x^3 + \frac{x^4}{12a^2}
 \end{aligned}$$

Make $\dot{y}=0$, &c. then assuming any constant quantity a , it may be $y=a$, &c. Then by Division $-y^{-1} = -a^{-1}$, &c. and therefore $\dot{y} = * + a^{-1}$, &c. and consequently $y = * + a^{-1}x$, &c. Then by Division $-y^{-1} = * + a^{-3}x$, &c. and therefore $\dot{y} = * * - a^{-3}x$, &c. and consequently $y = * * - \frac{1}{2}a^{-3}x^2$, &c. Then again by Division $-y^{-1} = * * - \frac{3}{2}a^{-5}x^2$, &c. and therefore $\dot{y} = * * * + \frac{3}{2}a^{-5}x^2 - x^2$, &c. and consequently $y = * * * - \frac{1}{2}a^{-5}x^3 + \frac{1}{3}x^3$, &c. And so of the rest. Here if we make $a=1$, we shall have $y=1+x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4$, &c.

Or the same Equation may be thus resolved:

$$\begin{aligned}
 -y^{-1} & \left. \begin{aligned} &= -x^2 + 2x^{-3} + 14x^{-8} + 216x^{-13}, \&c. \\ &- 2x^{-3} - 14x^{-8} - 216x^{-13}, \&c. \\ &y = x^{-2} + 2x^{-7} + 18x^{-12} + 280x^{-17}, \&c. \end{aligned} \right\}
 \end{aligned}$$

Make $-y^{-1} = -x^2$, &c. or $y = x^{-2}$, &c. Then $\dot{y} = -2x^{-3}$, &c. and therefore $-y^{-1} = * + 2x^{-3}$, &c. and consequently by Division $y = * + 2x^{-7}$, &c. Then $\dot{y} = * - 14x^{-8}$, &c. and therefore $-y^{-1} = * * + 14x^{-8}$, &c. and by Division $y = * * + 18x^{-12}$, &c. Then $\dot{y} = * * - 216x^{-13}$, &c. and therefore $-y^{-1} = * * * + 216x^{-13}$, &c. and by Division $y = * * * + 280x^{-17}$, &c. And so on.

Another ascending Series may be had from this Equation, viz.

$$y = \sqrt{2x} - \frac{2}{7}x^{\frac{3}{2}} + \frac{x^{\frac{5}{2}}}{147\sqrt{2}} + \frac{10x^{\frac{7}{2}}}{17493}, \&c. \text{ by multiplying it by } y, \text{ and then making } 1 \text{ the first Term of } y\dot{y}.$$

44. The Equation $\dot{y} = 3 + 2y - x^{-1}y^2$ may be thus resolved:

$$\begin{aligned}
 \dot{y} & \left. \begin{aligned} &= 3 - 3x + 6x^2, \&c. \\ &- 6x + 3x^2, \&c. \\ &+ 9x - 9x^2, \&c. \\ &y = 3x - \frac{3}{2}x^2 + 2x^3, \&c. \end{aligned} \right\} \\
 -2y & \left. \begin{aligned} &= 3 - 3x + 6x^2, \&c. \\ &- 6x + 3x^2, \&c. \\ &+ 9x - 9x^2, \&c. \\ &y = 3x - \frac{3}{2}x^2 + 2x^3, \&c. \end{aligned} \right\} \\
 +x^{-1}y^2 & \left. \begin{aligned} &= 3 - 3x + 6x^2, \&c. \\ &- 6x + 3x^2, \&c. \\ &+ 9x - 9x^2, \&c. \\ &y = 3x - \frac{3}{2}x^2 + 2x^3, \&c. \end{aligned} \right\}
 \end{aligned}$$

Make

Make $\dot{y} = 3$, &c. then $y = 3x$, &c. Therefore $-2y = -6x$, &c. and $x^{-1}y^2 = 9x$, &c. and consequently $\dot{y} = * - 3x$, &c. Therefore $y = * - \frac{3}{2}x^2$, &c. Then $-2y = * + 3x^2$, &c. and $x^{-1}y^2 = * - 9x^2$, &c. Therefore $\dot{y} = * + 6x^2$, &c. and $y = * + 2x^3$, &c. &c.

Or the Resolution may be perform'd after these two following manners :

$$\begin{array}{l} -2y \\ +\dot{y} \\ +x^{-1}y^2 \end{array} \left. \begin{array}{l} = 3 - \frac{2}{4}x^{-1} + \frac{2}{2}x^{-2}, \&c. \\ = * - \frac{2}{8}x^{-2}, \&c. \\ = * + \frac{2}{4}x^{-1} - \frac{2}{8}x^{-2}, \&c. \end{array} \right\} \begin{array}{l} yx^{-1} \\ +\dot{y}y^{-1} \\ -3y^{-1} \end{array} \left. \begin{array}{l} = 2 + \frac{1}{2}x^{-1} - \frac{3}{8}x^{-2}, \&c. \\ = * + x^{-1} \\ = -\frac{3}{4}x^{-1} + \frac{3}{8}x^{-2}, \&c. \end{array} \right\} \begin{array}{l} y = -\frac{3}{2} + \frac{2}{8}x^{-1} - \frac{2}{4}x^{-2}, \&c. \\ y = 2x + \frac{1}{2} - \frac{3}{8}x^{-1}, \&c. \end{array}$$

Make $-2y = 3$, &c. or $y = -\frac{3}{2}$, &c. then $\dot{y} = 0$, &c. and $x^{-1}y^2 = +\frac{9}{4}x^{-1}$, &c. Therefore $-2y = * - \frac{2}{4}x^{-1}$, &c. or $y = * + \frac{2}{8}x^{-1}$, &c. and $\dot{y} = * - \frac{2}{8}x^{-2}$, &c. and by squaring $x^{-1}y^2 = * - \frac{2}{8}x^{-2}$, &c. and therefore $-2y = * + \frac{2}{2}x^{-2}$, &c. and $y = * - \frac{2}{4}x^{-2}$, &c. And so on.

Again, divide the whole Equation by y , and make $x^{-1}y = 2$, &c. then $y = 2x$, &c. And because $\dot{y} = 2$, &c. and $y^{-1} = \frac{1}{2}x^{-1}$, &c. 'tis $\dot{y}y^{-1} = x^{-1}$, &c. and $-3y^{-1} = -\frac{3}{2}x^{-1}$, &c. therefore $\dot{y}x^{-1} = * + \frac{1}{2}x^{-1}$, &c. and $y = * + \frac{1}{2}$, &c. Then because $\dot{y}y^{-1} = * + 0$, &c. and $-3y^{-1} = * + \frac{3}{8}x^{-2}$, &c. 'tis $yx^{-1} = * - \frac{3}{8}x^{-2}$, &c. and $y = * - \frac{3}{8}x^{-1}$, &c. &c.

45, 46. If the proposed Equation be $\dot{y} = -y + x^{-1} - x^{-2}$, its Solution may be thus :

$$\begin{array}{l} \dot{y} \\ +y \end{array} \left. \begin{array}{l} = -x^{-2} + x^{-1} \\ = -x^{-1} \\ = +x^{-1} \\ y = x^{-1} \end{array} \right\} \begin{array}{l} y \\ +\dot{y} \end{array} \left. \begin{array}{l} = x^{-1} - x^{-2} \\ = +x^{-2} \\ = -x^{-2} \\ y = x^{-1} \end{array} \right\} \begin{array}{c} x^0 \\ x^{-1} \\ x^{-2} \end{array} \left| \begin{array}{cc} y^1 & y^0 \\ -y & * \\ -\dot{y} & +x^{-1} \\ * & -x^{-2} \end{array} \right\} = 0.$$

Make $\dot{y} = -x^{-2}$, &c. then $y = x^{-1}$, &c. Consequently $\dot{y} = * 0$, &c. and therefore $y = * 0$, &c. that is, $y = x^{-1}$.

Again, make $y = x^{-1}$, &c. then $\dot{y} = -x^{-2}$, &c. and consequently $y = * + 0$, &c. that is, $y = x^{-1}$.

That this should be so, may appear by the direct Method. For if $y = x^{-1}$, 'tis $\dot{y} = -\dot{x}x^{-2}$; also $y\dot{x} = \dot{x}x^{-1}$. Then adding these two Equations together, 'tis $y\dot{x} + \dot{y} = \dot{x}x^{-1} - \dot{x}x^{-2}$, or $\dot{y} = -y + x^{-1} - x^{-2}$. Thus may we form as many Fluxional Equations as

as we please, of which the Fluents may be express'd in finite Terms; but to return to these again may sometimes require particular Expedients. Thus if we assume the Equation $y = 2x - \frac{4}{3}x^2 + \frac{1}{5}x^3$, taking the Fluxions, and putting $\dot{x} = 1$, we shall have $\dot{y} = 2 - \frac{8}{3}x + \frac{3}{5}x^2$, as also $\frac{y}{2x} = 1 - \frac{2}{3}x + \frac{1}{10}x^2$. Subtract this last from the foregoing Equation, and we shall have $\dot{y} - \frac{y}{2x} = 1 - 2x + \frac{1}{2}x^2$, the Solution of which here follows.

47. Let the propos'd Equation be $\dot{y} = \frac{y}{2x} + 1 - 2x + \frac{1}{2}x^2$, of which the Solution may be thus:

$$\begin{array}{l} \dot{y} \left\{ \begin{array}{l} = 1 - 2x + \frac{1}{2}x^2 \\ + e + fx + gx^2 \end{array} \right. \\ - \frac{y}{2x} \left\{ \begin{array}{l} - e - fx - gx^2 \end{array} \right. \\ \hline y = 2ex + 2fx^2 + 2gx^3 \\ y = 2x - \frac{4}{3}x^2 + \frac{1}{5}x^3 \end{array} \quad \begin{array}{l} \dot{y} \left\{ \begin{array}{l} = \frac{1}{2}x^2 - 2x + 1 \\ + ex^2 + fx + g \end{array} \right. \\ - \frac{y}{2x} \left\{ \begin{array}{l} - ex^2 - fx - g \end{array} \right. \\ \hline y = 2ex^3 + 2fx^2 + 2g \\ y = \frac{1}{5}x^3 - \frac{4}{3}x^2 + 2x \end{array} \quad \begin{array}{l} x^2 \left| \begin{array}{cc} y^1 & y^0 \\ * & +\frac{1}{2}x^2 \end{array} \right. \\ x^1 & * & -2x \\ x^0 & * & +1 \\ x^{-1} & +\frac{1}{2}yx^{-1} & * \end{array} \right\} = 0. \end{array}$$

By tabulating the Terms of this Equation, as usual, it may be observed, that one of the external Terms $-\dot{y} + \frac{1}{2}yx^{-1}$ is a double Term, to which the other external Term 1 belongs in common. Therefore to separate these, assume $y = 2ex$, &c. then $-\frac{y}{2x} = -e$, &c. and consequently $\dot{y} = 1 + e$, &c. and therefore $y = x + ex$, &c. That is, because $2ex = x + ex$, or $2e = 1 + e$, 'tis $e = 1$, or $y = 2x$, &c. So if we make $y = * + 2fx^2$, &c. then $-\frac{y}{2x} = * - fx$, &c. therefore $\dot{y} = * + fx - 2x$, &c. and $y = * + \frac{1}{2}fx^2 - x^2$, &c. that is, $2f = \frac{1}{2}f - 1$, or $f = -\frac{2}{3}$. So that $y = * - \frac{4}{3}x^2$, &c. So if we make $y = * * + 2gx^3$, &c. then $-\frac{y}{2x} = * * - gx^2$, &c. and therefore $\dot{y} = * * + gx^2 + \frac{1}{2}x^2$, &c. and $y = * * + \frac{1}{3}gx^3 + \frac{1}{6}x^3$, &c. or $2g = \frac{1}{3}g + \frac{1}{6}$, or $g = \frac{1}{10}$, so that $y = * * \frac{1}{5}x^3$, &c. So if we make $y = * * * + 2hx^4$, &c. then $-\frac{y}{2x} = * * * - hx^3$, &c. and therefore $\dot{y} = * * * + hx^3$, &c. and $y = * * * + \frac{1}{4}hx^4$, &c. But because here $2h = \frac{1}{4}h$, this Equation would be absurd except $h = 0$. And so all the subsequent Terms will vanish *in infinitum*, and this will be the exact value of y . And the same may be done from the other case of external Terms, as will appear from the Paradigm.

48. Nothing can be added to illustrate this Investigation, unless we would demonstrate it synthetically. Because $y = ex^3$, as is here found,

found, therefore $\dot{y} = \frac{3}{4}ex^{\frac{3}{4}-1}$, or $\dot{y} = \frac{3ex^{\frac{3}{4}}}{4x}$. Here instead of $ex^{\frac{3}{4}}$ substitute y , and we shall have $\dot{y} = \frac{3y}{4x}$, as given at first.

49, 50. The given Equation $\dot{y} = yx^{-2} + x^{-2} + 3 + 2x - 4x^{-1}$ may be thus resolved after a general manner.

$$\begin{aligned} y \left\{ \begin{aligned} &= 2x + 3 - 4x^{-1} + x^{-2} - x^{-3} + \frac{1}{2}x^{-4}, \&c. \\ &+ 1 + 4x^{-1} + ax^{-2} - ax^{-3} + \frac{1}{2}ax^{-4} \\ -x^{-2}y \left\{ \begin{aligned} &- - - - - 1 - 4x^{-1} - ax^{-2} + x^{-3} - \frac{1}{2}x^{-4}, \&c. \\ &+ ax^{-3} - \frac{1}{2}ax^{-4} \end{aligned} \right. \end{aligned} \right. \\ y = x^2 + 4x + a - x^{-1} + \frac{1}{2}x^{-2} - \frac{1}{6}x^{-3}, \&c. \\ -ax^{-1} + \frac{1}{2}ax^{-2} - \frac{1}{6}ax^{-3}. \end{aligned}$$

Make $\dot{y} = 2x$, &c. then $y = x^2$, &c. Therefore $-x^{-2}y = -1$, &c. consequently $\dot{y} = * + 1 + 3$, &c. $= * + 4$, &c. and therefore $y = * + 4x$, &c. Then $-x^{-2}y = * - 4x^{-1}$, &c. and consequently $\dot{y} = ** + 0$, &c. and therefore assuming any constant quantity a , it may be $y = ** + a$, &c. Then $-x^{-2}y = ** - ax^{-2}$, &c. and therefore $\dot{y} = *** + ax^{-2} + x^{-2}$, &c. and $y = *** - ax^{-1} - x^{-1}$, &c. And so on. Here if we make $a = 0$, 'tis $y = x^2 + 4x * - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3}$, &c.

51, 52. The Equation of this Example is $\dot{y} = 3xy^{\frac{2}{3}} + y$, which we shall resolve by our usual Method, without any other preparation than dividing the whole by $y^{\frac{2}{3}}$, that one of the Terms may be clear'd from the Relate Quantity; which will reduce it $\dot{y}y^{-\frac{2}{3}} = y^{\frac{1}{3}} = 3x$, of which the Resolution may be thus:

$$\begin{aligned} \dot{y}y^{-\frac{2}{3}} \left\{ \begin{aligned} &= 3x + \frac{1}{2}x^2 + \frac{1}{18}x^3 + \frac{1}{216}x^4 + \frac{1}{3240}x^5, \&c. \\ -y^{\frac{1}{3}} \left\{ \begin{aligned} &- - - - - \frac{1}{2}x^2 - \frac{1}{18}x^3 - \frac{1}{216}x^4 - \frac{1}{3240}x^5, \&c. \end{aligned} \right. \end{aligned} \right. \\ y = \frac{1}{8}x^6 + \frac{1}{24}x^7 + \frac{1}{288}x^8, \&c. \end{aligned}$$

Make $\dot{y}y^{-\frac{2}{3}} = 3x$, &c. or taking the Fluents, $3y^{\frac{1}{3}} = \frac{3}{2}x^2$, &c. or $y^{\frac{1}{3}} = \frac{1}{2}x^2$, &c. or $y = \frac{1}{8}x^6$, &c. And because $-y^{\frac{1}{3}} = -\frac{1}{2}x^2$, &c. it will be $\dot{y}y^{-\frac{2}{3}} = * + \frac{1}{2}x^2$, &c. and therefore $3y^{\frac{1}{3}} = * + \frac{1}{6}x^3$, &c. and $y^{\frac{1}{3}} = * + \frac{1}{18}x^3$, &c. and by cubing $y = * + \frac{1}{24}x^7$, &c. Then because $-y^{\frac{1}{3}} = * - \frac{1}{18}x^3$, &c. 'tis $\dot{y}y^{-\frac{2}{3}} = ** + \frac{1}{18}x^3$, &c. and therefore $3y^{\frac{1}{3}} = ** + \frac{1}{72}x^4$, &c. and $y^{\frac{1}{3}} = ** + \frac{1}{216}x^4$, &c. and by cubing $y = *** + \frac{1}{288}x^8$, &c. And so on.

53. Lastly, in the Equation $\dot{y} = 2y^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}$, or $\dot{y}y^{-\frac{1}{2}} = 2\dot{x} + \dot{x}x^{\frac{1}{2}}$, assuming c for a constant quantity, whose Fluxion therefore is 0, and taking the Fluents, it will be $2y^{\frac{1}{2}} = 2c + 2x + \frac{2}{3}x^{\frac{3}{2}}$, or $y^{\frac{1}{2}} = c + x + \frac{1}{3}x^{\frac{3}{2}}$. Then by squaring, $y = c^2 + 2cx + x^2 + \frac{2}{3}cx^{\frac{3}{2}} + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$. Here the Root y may receive as many different values, while x remains the same, as c can be interpreted different ways. Make $c = 0$, then $y = x^2 + \frac{2}{3}x^{\frac{5}{2}} + \frac{1}{9}x^3$.

The Author is pleas'd here to make an Excuse for his being so minute and particular, in discussing matters which, as he says, will but seldom come into practice; but I think any Apology of this kind is needless, and we cannot be too minute, when the perfection of a Method is concern'd. We are rather much obliged to him for giving us his whole Method, for applying it to all the cases that may happen, and for obviating every difficulty that may arise. The use of these Extractions is certainly very extensive; for there are no Problems in the inverse Method of Fluxions, and especially such as are to be answer'd by infinite Series, but what may be reduced to such Fluxional Equations, and may therefore receive their Solutions from hence. But this will appear more fully hereafter.

SECT. IV. *Solution of the third Case of Equations, with some necessary Demonstrations.*

54. **F**OR the more methodical Solution of what our Author calls *a most troublesome and difficult Problem*, (and surely the Inverse Method of Fluxions, in its full extent, deserves to be call'd such a Problem,) he has before distributed it into three Cases. The first Case, in which two Fluxions and only one flowing Quantity occur in the given Equation, he has dispatch'd without much difficulty, by the assistance of his Method of infinite Series. The second Case, in which two flowing Quantities and their Fluxions are any how involved in the given Equation, even with the same assistance is still an operose Problem, but yet is discuss'd in all its varieties, by a sufficient number of apposite Examples. The third Case, in which occur more than two Fluxions with their Fluents, is here very artfully managed, and all the difficulties of it are reduced to the other two Cases. For if the Equation involves (for instance) three Fluxions, with some or all of their Fluents, another Equation ought to be given by the Question, in order to a full Determination,

termination, as has been already argued in another place; or if not, the Question is left indetermin'd, and then another Equation may be assumed *ad libitum*, such as will afford a proper Solution to the Question. And the rest of the work will only require the two former Cases, with some common Algebraic Reductions, as we shall see in the Author's Example.

55. Now to consider the Author's Example, belonging to this third Case of finding Fluents from their Fluxions given, or when there are more than two variable Quantities, and their Fluxions, either express'd or understood in the given Equation. This Example is $2\dot{x} - \dot{z} + y\dot{x} = 0$, in which because there are three Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, (and therefore virtually three Fluents x , y , and z ,) and but one Equation given; I may assume (for instance) $x = y$, whence $\dot{x} = \dot{y}$, and by substitution $2\dot{y} - \dot{z} + y\dot{y} = 0$, and therefore $2y - z + \frac{1}{2}y^2 = 0$. Now as here are only two Equations $x - y = 0$ and $2y - z + \frac{1}{2}y^2 = 0$, the Quantities x , y , and z are still variable Quantities, and susceptible of infinite values, as they ought to be. Indeed a third Equation may be had, as $2x - z + \frac{1}{2}x^2 = 0$; but as this is only derived from the other two, it brings no new limitation with it, but leaves the quantities still flowing and indeterminate quantities. Thus if I should assume $2y = a + z$ for the second Equation, then $2\dot{y} = \dot{z}$, and by substitution $2\dot{x} - 2\dot{y} + y\dot{x} = 0$, or $\dot{y} = \frac{2\dot{x}}{2-x} = \dot{x} + \frac{1}{2}x\dot{x} + \frac{1}{4}x^2\dot{x}$, &c. and therefore $y = x + \frac{1}{4}x^2 + \frac{1}{12}x^3$, &c. which two Equations are a compleat Determination. Again, if we assume with the Author $x = y^2$, and thence $\dot{x} = 2y\dot{y}$, we shall have by substitution $4y\dot{y} - \dot{z} + yy^2 = 0$, and thence $2y^2 - z + \frac{1}{3}y^3 = 0$, which two Equations are a sufficient Determination. We may indeed have a third, $2x - z + \frac{1}{3}x^{\frac{3}{2}} = 0$; but as this is included in the other two, and introduces no new limitation, the quantities will still remain fluent. And thus an infinite variety of second Equations may be assumed, tho' it is always convenient, that the assumed Equation should be as simple as may be. Yet some caution must be used in the choice, that it may not introduce such a limitation, as shall be inconsistent with the Solution. Thus if I should assume $2x - z = 0$ for the second Equation, I should have $2\dot{x} - \dot{z} = 0$ to be substituted, which would make $y\dot{x} = 0$, and therefore would afford no Solution of the Equation.

'Tis easy to extend this reasoning to Equations, that involve four or more Fluxions, and their flowing Quantities; but it would be needless here to multiply Examples. And thus our Author has completely solved this Case also, which at first view might appear formidable.

midable enough, by reducing all its difficulties to the two former Cases.

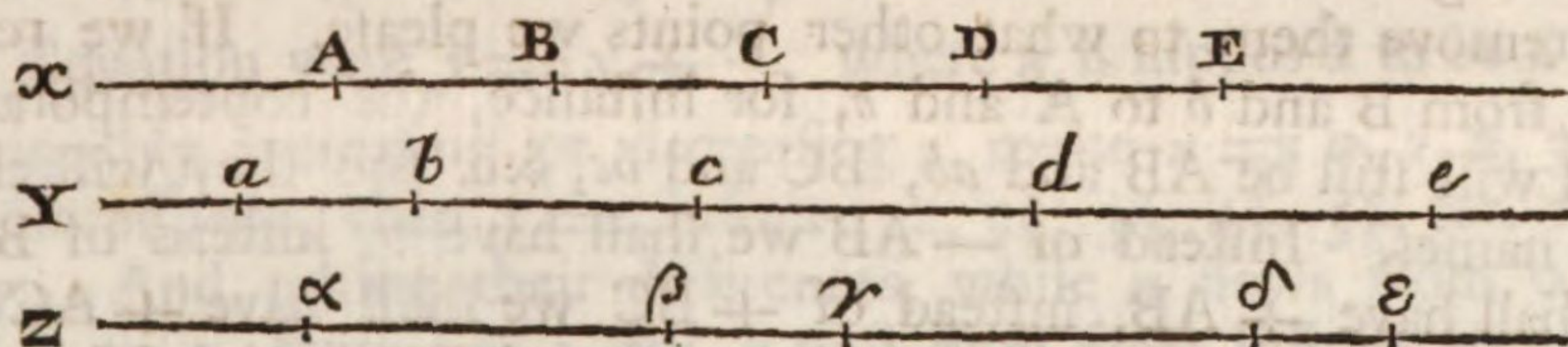
56, 57. The Author's way of demonstrating the Inverse Method of Fluxions is short, but satisfactory enough. We have argued elsewhere, that from the Fluents given to find the Fluxions, is a direct and synthetical Operation; and on the contrary, from the Fluxions given to find the Fluents, is indirect and analytical. And in the order of nature Synthesis should always precede Analysis, or Composition should go before Resolution. But the Terms Synthesis and Analysis are often used in a vague sense, and taken only relatively, as in this place. For the direct Method of Fluxions being already demonstrated synthetically, the Author declines (for the reasons he gives) to demonstrate the Inverse Method synthetically also, that is, primarily, and independently of the direct Method. He contents himself to prove it analytically, that is, by supposing the direct Method, as sufficiently demonstrated already, and shewing the necessary connexion between this and the inverse Method. And this will always be a full proof of the truth of the conclusions, as Multiplication is a good proof of Division. Thus in the first Example we found, that if the given Equation is $y + xy - \dot{y} = 3x - x^2 - 1$, we shall have the Root $y = x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6$, &c. To prove the truth of which conclusion, we may hence find, by the direct Method, $\dot{y} = 1 - 2x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 - \frac{2}{15}x^5$, &c. and then substitute these two Series in the given Equation, as follows:

$$\begin{array}{r}
 y \text{ --- } + x - x^2 + \frac{1}{3}x^3 - \frac{1}{6}x^4 + \frac{1}{30}x^5 - \frac{1}{45}x^6, \text{ \&c.} \\
 + xy \text{ --- } + x^2 - x^3 + \frac{1}{3}x^4 - \frac{1}{6}x^5 + \frac{1}{30}x^6, \text{ \&c.} \\
 - \dot{y} \text{ --- } 1 + 2x - x^2 + \frac{2}{3}x^3 - \frac{1}{6}x^4 + \frac{2}{15}x^5 - \frac{1}{90}x^6, \text{ \&c.} \\
 \hline
 = -1 + 3x - x^2 \quad * \quad * \quad * \quad *
 \end{array}$$

Now by collecting these Series, we shall find the result to produce the given Equation, and therefore the preceding Operation will be sufficiently proved.

58. In this and the subsequent paragraphs, our Author comes to open and explain some of the chief Mysteries of Fluxions and Fluents, and to give us a Key for the clearer apprehension of their nature and properties. Therefore for the Learners better instruction, I shall not think much to inquire something more circumstantially into this matter. In order to which let us conceive any number of right Lines, AE, ae, æe, &c. indefinitely extended both ways, along which a Body, or a describing Point, may be supposed to move in each Line,

Line, from the left-hand towards the right, according to any Law or Rate of Acceleration or Retardation whatever. Now the Motion of every one of these Points, at all times, is to be estimated by its distance from some fixt point in the same Line; and any such Points may be chosen for this purpose, in each Line, suppose B, b , β , in which all the Bodies have been, are, or will be, in the same Moment of Time, from whence to compute their contemporaneous Augments, Differences, or flowing Quantities. These Fluents may be conceived as negative before the Body arrives at that point, as nothing when in it, and affirmative when they are got beyond it. In the first Line AE, whose Fluent we denominate by x , we may suppose the Body to move uniformly, or with any equable Velocity; then may the Fluent x , or the Line which is continually described,



represent Time, or stand for the Correlate Quantity, to which the several Relate Quantities are to be constantly refer'd and compared. For in the second Line ae , whose Fluent we call y , if we suppose the Body to move with a Motion continually accelerated or retarded, according to any constant Rate or Law, (which Law is express'd by any Equation compos'd of x and y and known quantities;) then will there always be contemporaneous parts or augments, described in the two Lines, which parts will make the whole Fluents to be contemporaneous also, and accommodate themselves to the Equation in all its Circumstances. So that whatever value is assumed for the Correlate x , the corresponding or contemporaneous value of the Relate y may be known from the Equation, and *vice versa*. Or from the Time being given, here represented by x , the Space represented by y may always be known. The Origin (as we may call it) of the Fluent x is mark'd by the point B, and the Origin of the Fluent y by the point b . If the Bodies at the same time are found in A and a , then will the contemporaneous Fluents be $-BA$ and $-ba$. If at the same time, as was supposed, they are found in their respective Origins B and b , then will each Fluent be nothing. If at the same time they are found in C and c , then will their Fluents be $+BC$ and $+bc$. And the like of all other points, in which the

I

moving

moving Bodies either have been, or shall be found, at the same time.

As to the Origins of these Fluents, or the points from whence we begin to compute them, (for tho' they must be conceived to be variable and indetermin'd in respect of one of their Limits, where the describing points are at present, yet they are fixt and determin'd as to their other Limit, which is their Origin,) tho' before we appointed the Origin of each Fluent to be in B and b , yet it is not of absolute necessity that they should begin together, or at the same Moment of Time. All that is necessary is this, that the Motions may continue as before, or that they may observe the same rate of flowing, and have the same contemporaneous Increments or Decrements, which will not be at all affected by changing the beginnings of the Fluents. The Origins of the Fluents are intirely arbitrary things, and we may remove them to what other points we please. If we remove them from B and b to A and c , for instance, the contemporaneous Lines will still be AB and ab , BC and bc , &c. tho' they will change their names. Instead of $-AB$ we shall have 0 , instead of B or 0 we shall have $+AB$, instead of $+BC$ we shall have $+AC$; &c. So instead of $-ab$ we shall have $-ac + bc$, instead of b or 0 we shall have $-bc$, instead of $+bd$ we shall have $+bc + cd$, &c. That is, in the Equation which determines the general Law of flowing or increas'g, we may always increase or diminish x , or y , or both, by any given quantity, as occasion may require, and yet the Equation that arises will still express the rate of flowing; which is all that is necessary here. Of the use and conveniency of which Reduction we have seen several instances before.

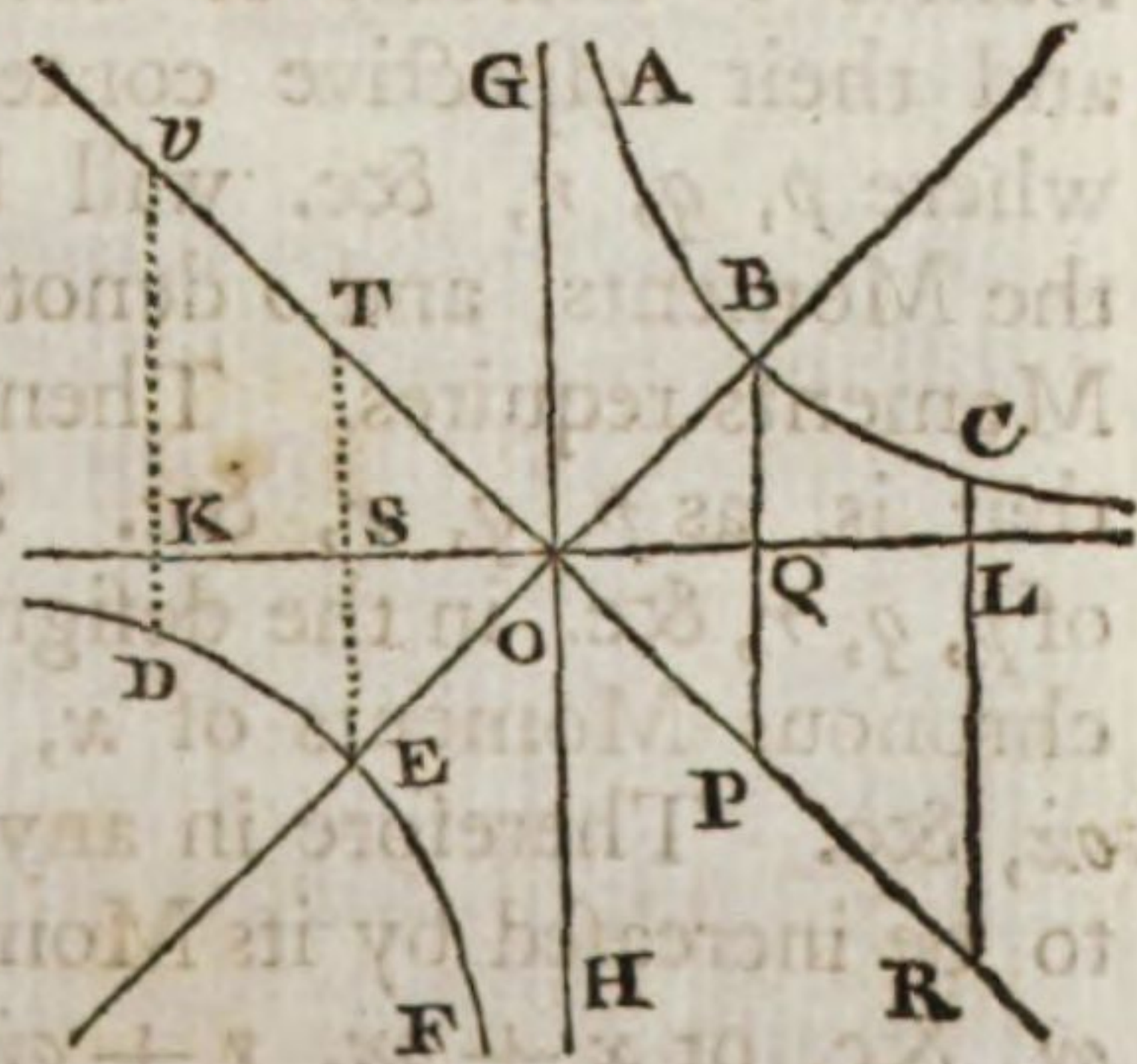
If there be a third Line $\alpha\epsilon$, described in like manner, whose Fluent may be z , having its parts corresponding with the others, as $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, &c. there must be another Equation, either given or assumed, to ascertain the rate of flowing, or the relation of z to the Correlate x . Or it will be the same thing, if in the two Equations the Fluents x , y , z , are any how promiscuously involved. For these two Equations will limit and determine the Law of flowing in each Line. And we may likewise remove the Origin of the Fluent z to what point we please of the Line $\alpha\epsilon$. And so if there were more Lines, or more Fluents.

59. To exemplify what has been said by an easy instance. Thus instead of the Equation $\dot{y} = xxy$, we may assume $\dot{y} = xy + xxy$, where the Origin of x is changed, or x is diminish'd by Unity; for $1 + x$ is substituted instead of x . The lawfulness of which Reduction

duction may be thus proved from the Principles of Analyticks. Make $x = 1 + z$, whence $\dot{x} = \dot{z}$, which shews, that x and z flow or increase alike. Substitute these instead of x and $\dot{x}$ in the Equation $\dot{y} = xxy$, and it will become $\dot{y} = zy + \dot{z}zy$. This differs in nothing else from the assumed Equation $\dot{y} = xy + \dot{x}xy$, only that the Symbol x is changed into the Symbol z , which can make no real change in the argumentation. So that we may as well retain the same Symbols as were given at first, and, because $z = x - 1$, we may as well suppose x to be diminish'd by Unity.

60, 61. The Equation expressing the Relation of the Fluents will at all times give any of their contemporaneous parts; for assuming different values of the Correlate Quantity, we shall thence have the corresponding different values of the Relate, and then by subtraction we shall obtain the contemporary differences of each. Thus if the given Equation were $y = x + \frac{1}{x}$, where x is suppos'd to be a quantity equably increasing or decreasing; make $x = 0, 1, 2, 3, 4, 5$, &c. successively, then $y =$ infinite, $2, 2\frac{1}{2}, 3\frac{1}{3}, 4\frac{1}{4}, 5\frac{1}{5}$, &c. respectively. And taking their differences, while x flows from 0 to 1, from 1 to 2, from 2 to 3, &c. y will flow from infinite to 2, from 2 to $2\frac{1}{2}$, from $2\frac{1}{2}$ to $3\frac{1}{3}$, &c. that is, their contemporaneous parts will be 1, 1, 1, 1, &c. and infinite, $\frac{1}{2}, \frac{5}{6}, \frac{11}{6},$ &c. respectively. Likewise, if we go backwards, or if we make x negative, we shall have $x = 0, -1, -2,$ &c. which will make $y =$ infinite, $-2, -2\frac{1}{2},$ &c. so that the contemporaneous differences will be as before.

Perhaps it may make a stronger impression upon the Imagination, to represent this by a Figure. To the rectangular Asymptotes GOH and KOL let ABC and DEF be opposite Hyperbola's; bisect the Angle GOK by the indefinite right Line vOR, perpendicular to which draw the Diameter BOE, meeting the Hyperbola's in B and E, from whence draw BQP and EST, as also CLR and DKU parallel to GOH. Now if OL is made to represent the indefinite and equable quantity x in the Equation $y = x + \frac{1}{x}$,



then CR may represent y . For $CL = \frac{1}{OL} = \frac{1}{x}$, (supposing $BQ = OQ = 1$), and $LR = OL = x$; therefore $CR = LR + CL$,
R r or

or $y = x + \frac{1}{x}$. Now the Origin of OL, or x , being in O; if $x = 0$, then CR, or y , will coincide with the Asymptote OG, and therefore will be infinite. If $x = 1 = OQ$, then $y = BP = 2$. If $x = 2 = OL$, then $y = CR = 2\frac{1}{2}$. And so of the rest. Also proceeding the contrary way, if $x = 0$, then y may be supposed to coincide with the Asymptote OH, and therefore will be negative and infinite. If $x = OS = -1$, then $y = ET = -2$. If $x = OK = -2$, then $y = Dv = -2\frac{1}{2}$, &c. And thus we may pursue, at least by Imagination, the correspondent values of the flowing quantities x and y , as also their contemporary differences, through all their possible varieties; according to their relation to each other, as exhibited by the Equation $y = x + \frac{1}{x}$.

The Transition from hence to Fluxions is so very easy, that it may be worth while to proceed a little farther. As the Equation expressing the relation of the Fluents will give (as now observed) any of their contemporary parts or differences; so if these differences are taken very small, they will be nearly as the Velocities of the moving Bodies, or points, by which they are described. For Motions continually accelerated or retarded, when perform'd in very small spaces, become nearly equable Motions. But if those differences are conceived to be diminished *in infinitum*, so as from finite differences to become Moments, or vanishing Quantities, the Motions in them will be perfectly equable, and therefore the Velocities of their Description, or the Fluxions of the Fluents, will be accurately as those Moments. Suppose then x, y, z , &c. to represent Fluents in any Equation, or Equations, and their Fluxions, or Velocities of increase or decrease, to be represented by $\dot{x}, \dot{y}, \dot{z}$, &c. and their respective contemporary Moments to be ox, oy, oz , &c. where p, q, r , &c. will be the Exponents of the Proportions of the Moments, and o denotes a vanishing quantity, as the nature of Moments requires. Then $\dot{x}, \dot{y}, \dot{z}$, &c. will be as ox, oy, oz , &c. that is, as p, q, r , &c. So that $\dot{x}, \dot{y}, \dot{z}$, &c. may be used instead of p, q, r , &c. in the designation of the Moments. That is, the synchronous Moments of x, y, z , &c. may be represented by ox, oy, oz , &c. Therefore in any Equation the Fluent x may be supposed to be increased by its Moment ox , and the Fluent y by its Moment oy , &c. or $x + ox, y + oy$, &c. may be substituted in the Equation instead of x, y , &c. and yet the Equation will still be true, because the Moments are supposed to be synchronous. From which Operation

ration an Equation will be form'd, which, by due Reduction, must necessarily exhibit the relation of the Fluxions.

Thus, for example, if the Equation $y = x + z$ be given, by Substitution we shall have $y + oy = x + ox + z + oz$, which, because $y = x + z$, will become $oy = ox + oz$, or $y = x + z$, which is the relation of the Fluxions. Here again, if we assume $z = \frac{1}{x}$,

or $zx = 1$, by increasing the Fluents by their contemporary Moments, we shall have $z + oz \times x + ox = 1$, or $zx + ozx + oxz + oozx = 1$. Here because $zx = 1$, 'tis $ozx + oxz + oozx = 0$, or $zx + xz + oozx = 0$. But because $oozx$ is a vanishing Term in respect of the others, 'tis $zx + xz = 0$, or $z = -\frac{xz}{x} = -\frac{x}{x^2}$.

Now as the Fluxion of z comes out negative, 'tis an indication that as x increases z will decrease, and the contrary. Therefore in the Equation $y = x + z$, if $z = \frac{1}{x}$, or if the relation of the Fluents be $y = x + \frac{1}{x}$, then the relation of the Fluxions will be $y = x - \frac{x}{x^2}$.

And as before, from the Equation $y = x + \frac{1}{x}$ we derived the contemporaneous parts, or differences of the Fluents; so from the Fluxional Equation $y = x - \frac{x}{x^2}$ now found, we may observe the rate of flowing, or the proportion of the Fluxions at different values of the Fluents.

For because it is $\dot{x} : \dot{y} :: 1 : 1 - \frac{1}{x^2} :: x^2 : x^2 - 1$; when $x = 0$, or when the Fluent is but beginning to flow, (consequently when y is infinite,) it will be $\dot{x} : \dot{y} :: 0 : -1$. That is, the Velocity wherewith x is described is infinitely little in comparison of the velocity wherewith y is described; and moreover it is insinuated, (because of -1), that while x increases by any finite quantity, tho' never so little, y will decrease by an infinite quantity at the same time. This will appear from the inspection of the foregoing Figure. When $x = 1$, (and consequently $y = 2$), then $\dot{x} : \dot{y} :: 1 : 0$. That is, x will then flow infinitely faster than y . The reason of which is, that y is then at its Limit, or the least that it can possibly be, and therefore in that place it is stationary for a moment, or its Fluxion is nothing in comparison of that of x . So in the foregoing Figure, BP is the least of all such Lines as are represented by CR. When $x = 2$, (and therefore $y = 2\frac{1}{2}$), it will be $\dot{x} : \dot{y} :: 4 : 3$. Or

the Velocity of x is there greater than that of y , in the ratio of 4 to 3. When $x = 3$, then $\dot{x} : \dot{y} :: 9 : 8$. And so on. So that the Velocities or Fluxions constantly tend towards equality, which they do not attain till $\frac{1}{x}$ (or CL) finally vanishing, x and y become equal. And the like may be observed of the negative values of x and y .

SECT. V. *The Resolution of Equations, whether Algebraical or Fluxional, by the assistance of superior orders of Fluxions.*

ALL the foregoing Extractions (according to a hint of our Author's,) may be perform'd something more expeditiously, and without the help of subsidiary Operations, if we have recourse to superior orders of Fluxions. To shew this first by an easy Instance.

Let it be required to extract the Cube-root of the Binomial $a^3 + x^3$, or to find the Root y of this Equation $y^3 = a^3 + x^3$; or rather, for simplicity-sake, let it be $y^3 = a^3 + z$. Then $y = a$, &c. or the initial Term of y will be a . Taking the Fluxions of this Equation, we shall have $3yy^2 = \dot{z} = 1$, or $\dot{y} = \frac{1}{3}y^{-2}$. But as it is $y = a$, &c. by substitution it will be $\dot{y} = \frac{1}{3}a^{-2}$, &c. and taking the Fluents, 'tis $y = * + \frac{1}{3}a^{-2}z$, &c. Here a vacancy is left for the first Term of y , which we already know to be a . For another Operation take the Fluxions of the Equation $\dot{y} = \frac{1}{3}y^{-2}$; whence $\ddot{y} = -\frac{2}{3}\dot{y}y^{-3} = -\frac{2}{9}y^{-5}$. Then because $y = a$, &c. 'tis $\ddot{y} = -\frac{2}{9}a^{-5}$, &c. and taking the Fluents, 'tis $\dot{y} = * - \frac{2}{9}a^{-5}z$, &c. and taking the Fluents again, 'tis $y = * * - \frac{1}{9}a^{-5}z^2$, &c. Here two vacancies are to be left for the two first Terms of y , which are already known. For the next Operation take the Fluxions of the Equation $\ddot{y} = -\frac{2}{9}y^{-5}$, that is, $\dddot{y} = +\frac{10}{9}\dot{y}y^{-6} = +\frac{10}{27}y^{-8}$. Or because $y = a$, &c. 'tis $\dddot{y} = \frac{10}{27}a^{-8}$, &c. Then taking the Fluents, 'tis $\ddot{y} = * \frac{10}{27}a^{-8}z$, &c. $\dot{y} = * * \frac{5}{27}a^{-8}z^2$, &c. and $y = * * * \frac{5}{81}a^{-8}z^3$, &c. Again, for another Operation take the Fluxions of the Equation $\dot{y} = \frac{1}{3}y^{-2}$; whence $\ddot{y} = -\frac{2}{3}\dot{y}y^{-3} = -\frac{2}{27}y^{-11}$. Or because $y = a$, &c. 'tis $\ddot{y} = -\frac{2}{27}a^{-11}$, &c. Then taking the Fluents, $\dot{y} = * - \frac{2}{27}a^{-11}z$, &c. $y = * * - \frac{1}{405}a^{-11}z^2$,

$\frac{4}{81}a^{-11}x^2$, &c. $y = \dots - \frac{4}{243}a^{-11}x^3$, &c. and $y = \dots - \frac{1}{243}a^{-11}x^4$, &c. And so we may go on as far as we please. We

have therefore found at last, that $y = a + \frac{x}{3a^2} - \frac{x^2}{9a^3} + \frac{5x^3}{81a^4} - \frac{10x^4}{243a^5}$, &c. or for x writing x^3 , 'tis $\sqrt[3]{a^3 + x^3} = a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8} - \frac{10x^{12}}{243a^{11}}$, &c.

Or universally, if we would resolve $a + x \mid^m$ into an equivalent infinite Series, make $y = a + x \mid^m$, and we shall have a^m for the first Term of the Series y , or it will be $y = a^m$, &c. Then because $y^{\frac{1}{m}} = a + x$, taking the Fluxions we shall have $\frac{1}{m}y^{\frac{1}{m}-1} =$

$\dot{x} = 1$, or $\dot{y} = my^{\frac{1}{m}-1}$. But because it is $y = a^m$, &c. it will be $\dot{y} = ma^{m-1}$, &c. and now taking the Fluents, 'tis $y = \frac{1}{m}a^m x$, &c. Again, because it is $\dot{y} = my^{\frac{1}{m}-1}$, taking the Fluxions it will be $\ddot{y} = m - 1 \dot{y} y^{\frac{1}{m}-2} = m \times m - 1 y^{\frac{1}{m}-2}$; and because $y = a^m$, &c. 'tis $\ddot{y} = m \times m - 1 a^{m-2}$, &c. And taking the Fluents, 'tis $\dot{y} = \frac{1}{2} m \times m - 1 a^{m-2} x$, &c. and therefore $y = \frac{1}{2} m \times m - 1 a^{m-2} x^2$, &c.

Again, because it is $\ddot{y} = m \times m - 1 y^{\frac{1}{m}-2}$, taking the Fluxions it will be $\dddot{y} = m - 1 \times m - 2 \dot{y} y^{\frac{1}{m}-3} = m \times m - 1 \times m - 2 y^{\frac{1}{m}-3}$; and because $y = a^m$, &c. 'tis $\dddot{y} = m \times m - 1 \times m - 2 a^{m-3}$, &c. And taking the Fluents, 'tis $\ddot{y} = \frac{1}{2} m \times m - 1 \times m - 2 a^{m-3} x$, &c. $\dot{y} = \frac{1}{6} m \times m - 1 \times m - 2 a^{m-3} x^2$, &c. and $y = \frac{1}{6} m \times m - 1 \times \frac{m-2}{3} a^{m-3} x^3$, &c. And so we might proceed as far as we please, if the Law of Continuation had not already been sufficiently manifest. So that we shall have here $a + x \mid^m = a^m + \frac{1}{m} a^{m-1} x + \frac{1}{2} m \times m - 1 a^{m-2} x^2 + \frac{1}{6} m \times m - 1 \times \frac{m-2}{3} a^{m-3} x^3 + \frac{1}{24} m \times m - 1 \times \frac{m-2}{2} \times \frac{m-3}{3} a^{m-4} x^4$, &c.

⁴ This is a famous Theorem of our Author's, tho' discover'd by him after a very different manner of Investigation, or rather by Induction. It is commonly known by the name of his Binomial Theorem, because by its assistance any Binomial, as $a + x$, may be raised to any Power at pleasure, or any Root of it may be extracted. And it is obvious, that when m is interpreted by any integer

teger affirmative Number, the Series will break off, and become finite, at a number of Terms denominated by m . But in all other cases it will be an infinite Series, which will converge when x is less than a .

Indeed it can hardly be said, that this, or any other that is derived from the Method of Fluxions, is a strict Investigation of this Theorem. Because that Method itself is originally derived from the Method of raising Powers, at least integral Powers, and previously supposes the knowledge of the *Unciæ*, or the numeral Coefficients. However it may answer the intention, of being a proper Example of this Method of Extraction, which is all that is necessary here.

There is another Theorem for this purpose, which I found many years ago, and then communicated it to my ingenious Friend Mr. *A. de Moivre*, who liked it so well as to insert it in a Mathematical Treatise he was then publishing. I shall here give the Reader its Investigation, in the same manner it was found.

Let us suppose $\overline{a+x}^m = a^m + p$, and that $a+x = z$, and therefore $\dot{z} = \dot{x} = 1$. Now because $z^m = a^m + p$, it will be $\dot{p} = m z^{m-1} = \frac{m z^m}{z}$; where for z^m writing its value $a^m + p$, we shall have $\dot{p} = \frac{m a^m}{z} + \frac{m p}{z}$. Now if we make $p = \frac{m a^m x}{z} + q$, it will be $\dot{p} = \frac{m a^m}{z} - \frac{m a^m x}{z^2} + \dot{q}$. And comparing these two values of $\dot{p}$, we shall have $\dot{q} = \frac{m a^m x}{z^2} + \frac{m p}{z}$; where if for p we write its value as above, it will be $\dot{q} = \frac{m a^m x}{z^2} + \frac{m^2 a^m x}{z^2} + \frac{m q}{z}$, or $\dot{q} = m \times \frac{m+1}{2} \times \frac{a^m x}{z^2} + \frac{m q}{z}$; make $q = m \times \frac{m+1}{2} \times \frac{a^m x^2}{z^2} + r$; therefore $\dot{q} = m \times \frac{m+1}{2} \times \frac{a^m x}{z^2} - m \times \frac{m+1}{2} \times \frac{a^m x^2}{z^3} + \dot{r}$. From which two values of $\dot{q}$ we shall have $\dot{r} = m \times \frac{m+1}{2} \times \frac{a^m x^2}{z^3} + \frac{m q}{z}$. And for q substituting its value, it will be $\dot{r} = m \times \frac{m+1}{2} \times \frac{a^m x^2}{z^3} + m^2 \times \frac{m+1}{2} \times \frac{a^m x^2}{z^3} + \frac{m r}{z}$. Or $\dot{r} = m \times \frac{m+1}{2} \times \frac{m+2}{1} \times \frac{a^m x^2}{z^3} + \frac{m r}{z}$. Make $r = m \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{a^m x^3}{z^3} + s$; then, &c. So that we shall

shall have $\overline{a+x}^m = a^m + m \times \frac{a^m x}{a+x} + m \times \frac{m+1}{2} \times \frac{a^m x^2}{a+x|^2} + m \times \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{a^m x^3}{a+x|^3}, \&c.$

Now this Series will stop of its own accord, at a finite number of Terms, when m is any integer and negative Number; that is, when the Reciprocal of any Power of a Binomial is to be found. But in all other cases we shall have an infinite converging Series for the Power or Root required, which will always converge when a and x have the same Sign; because the Root of the Scale, or the converging quantity, is $\frac{x}{a+x}$, which is always less than Unity.

By comparing these two Series together, or by collecting from each the common quantity $\frac{\overline{a+x}^m - a^m}{ma^m x}$, we shall have the two equivalent Series $\frac{1}{a} + \frac{m-1}{2} \times \frac{x}{a^2} + \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{x^2}{a^3}, \&c. = \frac{1}{a+x} + \frac{m+1}{2} \times \frac{x}{a+x|^2} + \frac{m+1}{2} \times \frac{m+2}{3} \times \frac{x^2}{a+x|^3}, \&c.$ from whence we might derive an infinite number of Numeral Converging Series, not inelegant, which would be proper to explain and illustrate the nature of Convergency in general, as has been attempted in the former part of this work. For if we assume such a value of m as will make either of the Series become finite, the other Series will exhibit the quantity that arises by an Approximation *ad infinitum*. And then a and x may be afterwards determined at pleasure.

As another Example of this Method, we shall shew (according to promise) how to derive Mr. *de Moivre's* elegant Theorem; for raising an Infinitinomial to any indeterminate Power, or for extracting any Root of the same. The way how it was derived from the abstract consideration of the nature and genesis of Powers, (which indeed is the only legitimate method of Investigation in the present case,) and the Law of Continuation, have been long ago communicated and demonstrated by the Author, in the Philosophical Transactions, N^o 230. Yet for the dignity of the Problem, and the better to illustrate the present Method of Extraction of Roots, I shall deduce it here as follows.

Let us assume the Equation $\overline{a + bx + cx^2 + dx^3 + ex^4, \&c.}^m = y$, where the value of y is to be found by an infinite Series, of which the first Term is already known to be a^m , or it is $y = a^m, \&c.$ Make $v = a + bx + cx^2 + dx^3 + ex^4, \&c.$ and putting $z = 1$, and taking the Fluxions, we shall have $\dot{v} = b + 2cz + 3dz^2$

$3dz^2 + 4ez^3$, &c. Then because $y = v^m$, it is $\dot{y} = m\dot{v}v^{m-1}$, where if we make $v = a$, &c. and $\dot{v} = b$, &c. we shall have $\dot{y} = ma^{m-1}b$, &c. and taking the Fluents, it will be $y = * ma^{m-1}bz$, &c.

For another Operation, because $\dot{y} = m\dot{v}v^{m-1}$, it is $\ddot{y} = m\ddot{v}v^{m-1} + m \times m - 1 \dot{v}^2 v^{m-2}$. And because $\ddot{v} = 2c + 6dz + 12ez^2$, &c. for v , $\dot{v}$, and $\ddot{v}$ substituting their values a , &c. b , &c. and $2c$, &c. respectively, we shall have $\ddot{y} = 2mca^{m-1} + m \times m - 1 b^2 a^{m-2}$, &c. and taking the Fluents $\dot{y} = * 2mca^{m-1}z + m \times m - 1 b^2 a^{m-2}z$, &c. and taking the Fluents again, $y = * * mca^{m-1}z^2 + m \times \frac{m-1}{2} b^2 a^{m-2}z^2$, &c.

For another Operation, because $\ddot{y} = m\ddot{v}v^{m-1} + m \times m - 1 \dot{v}^2 v^{m-2}$, 'tis $\ddot{\ddot{y}} = m\ddot{\ddot{v}}v^{m-1} + 3m \times m - 1 \ddot{v}\dot{v}v^{m-2} + m \times m - 1 \times m - 2 \dot{v}^3 v^{m-3}$. And because $\ddot{\ddot{v}} = 6d + 24ez$, &c. for v , $\dot{v}$, $\ddot{v}$, $\ddot{\ddot{v}}$, substituting a , &c. b , &c. $2c$, &c. $6d$, &c. we shall have $\ddot{\ddot{y}} = 6mda^{m-1} + 6m \times m - 1 bca^{m-2} + m \times m - 1 \times m - 2 b^3 a^{m-3}$, &c. And taking the Fluents it will be $\ddot{y} = * 6mda^{m-1}z + 6m \times m - 1 bca^{m-2}z + m \times m - 1 \times m - 2 b^3 a^{m-3}z$, &c. $\dot{y} = * * 3mda^{m-1}z^2 + 3m \times m - 1 bca^{m-2}z^2 + m \times \frac{m-1}{2} \times m - 2 b^3 a^{m-3}z^2$, &c. and $y = * * * mda^{m-1}z^3 + m \times \frac{m-1}{1} bca^{m-2}z^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} b^3 a^{m-3}z^3$, &c. And so on in infinitum. We shall therefore have $a + bz + cz^2 + dz^3 + ez^4$, &c. $|^m = (y =) a^m + ma^{m-1}bz + m \times \frac{m-1}{2} a^{m-2}b^2z^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}b^3z^3$, &c. $+ ma^{m-1}c + m \times \frac{m-1}{1} a^{m-2}bc + ma^{m-1}d$

And if the whole be multiply'd by z^m , and continued to a due length, it will have the form of Mr. de Moivre's Theorem.

The Roots of all Algebraical or Fluential Equations may be extracted by this Method. For an Example let us take the Cubick Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$, so often before resolved, in which $y = a$, &c. Then taking the Fluxions, and making $\dot{x} = 1$, we shall have $3yy^2 + ay + ax\dot{y} + a^2\dot{y} - 3x^2 = 0$. Here if for y we substitute a , &c. we shall have $4a^2\dot{y} + a^2 + ax\dot{y} - 3x^2$, &c. $= 0$, or $\dot{y} = \frac{-a^2 + 3x^2}{4a^2 + ax}$, &c. $= \frac{-a^2}{4a^2}$, &c. $= -\frac{1}{4}$, &c. And taking the Fluents, $y = * -\frac{1}{4}x$, &c. Then taking the Fluxions again

again of the last Equation, we shall have $3\ddot{y}y^2 + 6\dot{y}^2y + 2a\dot{y} + ax\ddot{y} + a^2\ddot{y} - 6x = 0$. Where if we make $y = a$, &c. and $\dot{y} = -\frac{1}{4}$, &c. we shall have $\ddot{y} = \frac{-\frac{3}{8}a + \frac{1}{2}a, \&c.}{4a^2, \&c.} = \frac{1}{32a}$, &c. and therefore

$\dot{y} = * + \frac{x}{32a}$, &c. and $y = ** + \frac{x^2}{64a}$, &c. Again, $3\ddot{y}y^2 + 18\ddot{y}\dot{y}y$

$+ 6\dot{y}^3 + 3a\ddot{y} + ax\ddot{y} + a^2\ddot{y} - 6 = 0$. Make $y = a$, &c. $\dot{y} =$

$-\frac{1}{4}$, &c. and $\ddot{y} = \frac{1}{32a}$, &c. then $\ddot{y} = \frac{\frac{9}{4} + \frac{9}{4} - \frac{3}{2} + 6}{4a^2}$, &c. =

$\frac{393}{256a^2}$, &c. and therefore $\dot{y} = * \frac{393x}{256a^2}$, &c. $\ddot{y} = ** \frac{393x^2}{512a^2}$, &c.

and $y = *** \frac{131x^3}{512a^2}$, &c. Again, $3\ddot{y}y^2 + 24\ddot{y}\dot{y}y + 18\dot{y}^2y + 36\ddot{y}\dot{y}^2$

$+ 4a\ddot{y} + ax\ddot{y} + a^2\ddot{y} = 0$. Make $y = a$, &c. $\dot{y} = -\frac{1}{4}$, &c. $\ddot{y} =$

$\frac{1}{32a}$, &c. and $\ddot{y} = \frac{393}{256a^2}$, &c. then $\ddot{y} = -\frac{24\ddot{y}\dot{y}y + 18\dot{y}^2y + 36\ddot{y}\dot{y}^2 + 4a\ddot{y}}{3y^2 + a^2, \&c.}$

$= \frac{1527}{2048a^3}$, &c. and $\ddot{y} = * \frac{1527x}{2048a^3}$, &c. $\ddot{y} = ** \frac{1527x^2}{4096a^3}$, &c. $\ddot{y} = ***$

$\frac{509x^3}{4096a^3}$, &c. and $y = **** \frac{509x^4}{16384a^3}$, &c. And so on as far as we please.

Therefore the Root is $y = a - \frac{1}{4}x + \frac{x^2}{64a} + \frac{131x^3}{512a^2} + \frac{509x^4}{16384a^3}$, &c.

The Series for the Root, when found by this Method, must always have its Powers ascending; but if we desire likewise to find

a Series with descending Powers, it may be done by this easy artifice. As in the present Equation $y^3 + axy + a^2y - x^3 - 2a^3 = 0$,

we may conceive x to be a constant quantity, and a to be a flowing quantity; or rather, to prevent a confusion of Ideas, we may change

a into x , and x into a , and then the Equation will be $y^3 + axy + x^2y - a^3 - 2x^3 = 0$. In this we shall have $y = a$, &c. and taking

the Fluxions, 'tis $3\ddot{y}y^2 + ay + ax\ddot{y} + 2xy + x^2\ddot{y} - 6x^2 = 0$,

or $\ddot{y} = \frac{-ay - 2xy + 6x^2}{3y^2 + ax + x^2}$. But because $y = a$, &c. 'tis $\ddot{y} = \frac{-aa}{3aa}$, &c.

$= -\frac{1}{3}$, &c. and therefore $y = * - \frac{1}{3}x$, &c. Again taking the Fluxions

'tis $3\ddot{y}y^2 + 6\dot{y}^2y + 2a\dot{y} + ax\ddot{y} + 2y + 4x\dot{y} + x^2\ddot{y} - 12x = 0$, or

$\ddot{y} = \frac{-6\dot{y}^2y - 2a\dot{y} - 2y - 4x\dot{y} + 12x}{3y^2 + ax + x^2} = \frac{-6\dot{y}^2y - 2a\dot{y} - 2y}{3y^2}$, &c. Or making

$y = a$, &c. and $\dot{y} = -\frac{1}{3}$, &c. 'tis $\ddot{y} = \frac{-\frac{2}{3}a + \frac{2}{3}a - 2a}{3a^2}$, &c.

$= -\frac{2}{3a}$, &c. and $\dot{y} = * - \frac{2x}{3a}$, &c. and $y = ** - \frac{x^2}{3a}$, &c.

Again it is $3\ddot{y}y^2 + 18\ddot{y}\dot{y}y + 6\dot{y}^3 + 3a\ddot{y} + ax\ddot{y} + 6\dot{y} + 6x\ddot{y} + x^2\ddot{y}$
S f — 12

$-12 = 0$, or $\dot{y} = \frac{-18\ddot{y}y - 6y^3 - 3a\ddot{y} - 6y + 12}{3y^2}$, &c. = (by making $y = a$, &c. $\dot{y} = -\frac{1}{3}$, &c. and $\ddot{y} = -\frac{2}{3a}$, &c.) $-\frac{4 + \frac{2}{9} + 2 + 2 + 12}{3a^2}$, &c. $= \frac{110}{27a^2}$, &c. Then taking the Fluents, $\ddot{y} = * \frac{110x}{27a^2}$, &c. $\dot{y} = * * \frac{55x^2}{27a^2}$, &c. and $y = * * * \frac{55x^3}{81a^2}$, &c. And so on. Therefore we shall have $y = a - \frac{1}{3}x - \frac{x^2}{3a} + \frac{55x^3}{81a^2}$, &c. Or now we may again change x into a , and a into x ; then it will be $y = x - \frac{1}{3}a - \frac{a^2}{3x} + \frac{55a^3}{81x^2}$, &c. for the Root of the given Equation, as was found before, pag. 216, &c.

Also in the Solution of Fluxional Equations, we may proceed in the same manner. As if the given Equation were $a^2\dot{y} - a^2\dot{x} + x^2\dot{y} = 0$, (in which, if the Radius of a Circle be represented by a , and if y be any Arch of the same, the corresponding Tangent will be represented by x ;) let it be required to extract the Root y out of this Equation, or to express it by a Series composed of the Powers of a and x . Make $\dot{x} = 1$, then the Equation will be $a^2\dot{y} - a^2 + x^2\dot{y} = 0$. Here because $\dot{y} = \frac{a^2}{a^2 + x^2} = 1$, &c. taking the Fluents it will be $y = * x$, &c. Then taking the Fluxions of this Equation, we shall have $a^2\ddot{y} + 2x\dot{y} + x^2\ddot{y} = 0$, or $\ddot{y} = -\frac{2x\dot{y}}{a^2 + x^2}$. But because we are to have a constant quantity for the first Term of $\ddot{y}$, we may suppose $\ddot{y} = \frac{0 - 2x\dot{y}}{a^2 + x^2} = 0$, &c. Then taking the Fluents 'tis $\dot{y} = * 0$, &c. and $y = * * 0$, &c. Then taking the Fluxions again, 'tis $a^2\ddot{y} + 2\dot{y} + 4x\dot{y} + x^2\ddot{y} = 0$, or $\ddot{y} = \frac{-2\dot{y} - 4x\dot{y}}{a^2 + x^2}$. Here if for $\dot{y}$ and $\ddot{y}$ we write their values 1, &c. and 0, &c. we shall have $\ddot{y} = -\frac{2}{a^2}$, &c. whence $\ddot{y} = * -\frac{2x}{a^2}$, &c. $\dot{y} = * * -\frac{x^2}{a^2}$, &c. and $y = * * * -\frac{x^3}{3a^2}$, &c. Taking the Fluxions again, 'tis $a^2\ddot{y} + 6\dot{y} + 6x\dot{y} + x^2\ddot{y} = 0$, or $\ddot{y} = \frac{-6\dot{y} - 6x\dot{y}}{a^2 + x^2} = 0$, &c. Therefore $\dot{y} = * 0$, &c. $\ddot{y} = * * 0$, &c. $\dot{y} = * * * 0$, &c. $y = * * * * 0$, &c. Again, $a^2\ddot{y} + 12\dot{y} + 8x\dot{y} + x^2\ddot{y} = 0$, or $\ddot{y} = \frac{-12\dot{y} - 8x\dot{y}}{a^2 + x^2}$

$= + \frac{24}{a^4}$, &c. Then $\ddot{y} = * + \frac{24x}{a^4}$, &c. $\ddot{\dot{y}} = * * + \frac{12x^2}{a^4}$, &c.
 $\ddot{y} = * * * + \frac{4x^3}{a^4}$, &c. $\dot{y} = * * * * + \frac{x^4}{a^4}$, &c. and $y = * * * * *$

$+ \frac{x^5}{5a^4}$, &c. Again, $a^2\dot{y} + 20\dot{y} + 10x\dot{y} + x^2\dot{y} = 0$, whence $y =$
 $* * * * * 0$, &c. Again, $a^2\ddot{y} + 30\ddot{y} + 12x\ddot{y} + x^2\ddot{y} = 0$, or $\ddot{y} =$
 $-\frac{30\dot{y} + 12x\dot{y}}{a^2 + x^2} = -30 \times 24a^{-6}$, &c. Then $\dot{y} = * - \frac{24 \times 30x}{a^6}$, &c.

$\dot{y} = * * - \frac{12 \times 30x^2}{a^6}$, &c. $\ddot{y} = * * * - \frac{4 \times 30x^3}{a^6}$, &c. $\ddot{\dot{y}} = * * * *$
 $- \frac{30x^4}{a^6}$, &c. $\ddot{\ddot{y}} = * * * * - \frac{6x^5}{a^6}$, &c. $\ddot{\ddot{\dot{y}}} = * * * * * - \frac{x^6}{a^6}$,
 &c. and $y = * * * * * - \frac{x^7}{7a^6}$, &c. And so on. So that we
 have here $y = * x + 0x^2 - \frac{x^3}{3a^2} + 0x^4 + \frac{x^5}{5a^4}$, &c. that is, $y =$
 $x - \frac{x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6}$, &c.

This Example is only to shew the universality of this Method,
 and how we are to proceed in other like cases; for as to the Equa-
 tion itself, it might have been resolved much more simply and ex-
 peditiously, in the following manner. Because $\dot{y} = \frac{a^2}{a^2 + x^2}$, by
 Division it will be $\dot{y} = 1 - \frac{x^2}{a^2} + \frac{x^4}{a^4} - \frac{x^6}{a^6} + \frac{x^8}{a^8}$, &c. And ta-
 king the Fluents, $y = x - \frac{x^3}{3a^2} + \frac{x^5}{5a^4} - \frac{x^7}{7a^6} + \frac{x^9}{9a^8}$, &c.

In the same Equation $a^2\dot{y} - a^2\dot{x} + x^2\dot{y} = 0$, if it were requir'd
 to express x by y , (the Tangent by the Arch,) or if x were made
 the Relate, and y the Correlate, we might proceed thus. Make
 $\dot{y} = 1$, then $a^2 - a^2\dot{x} + x^2 = 0$, or $\dot{x} = 1 + \frac{x^2}{a^2} = 1$, &c.

Then $x = * y$, &c. And taking the Fluxions, 'tis $\ddot{x} = \frac{2x\dot{x}}{a^2} =$
 $\frac{2x}{a^2}$, &c. $= 0 + \frac{2x}{a^2}$, &c. whence $\dot{x} = * 0$, &c. and $x = * * 0$,
 &c. So that the Terms of this Series will be alternately deficient,
 and therefore we need not compute them. Taking the Fluxions

again, 'tis $\ddot{x} = \frac{2\dot{x}^2}{a^2} + \frac{2x\ddot{x}}{a^2} = \frac{2}{a^2}$, &c. Therefore $\ddot{x} = * \frac{2y}{a^2}$, &c.

$\ddot{x} = * * \frac{y^2}{a^2}$, &c. and $x = * * * \frac{y^3}{3a^2}$, &c. Again, $\ddot{\ddot{x}} = \frac{6\dot{x}\ddot{x}}{a^2} + \frac{2x\ddot{\ddot{x}}}{a^2}$,
 S f 2 and

and again, $\dot{x}^5 = \frac{6\ddot{x}^2}{a^2} + \frac{8\ddot{x}\dot{x}}{a^2} + \frac{2\ddot{x}\dot{x}}{a^2}$. Substituting 1, &c. and $\frac{2}{a^2}$, &c. for $\dot{x}$ and $\ddot{x}$, and also 0, &c. for $\ddot{x}$ and $\ddot{x}$, it will be $\dot{x}^5 = \frac{16}{a^4}$, &c. whence $\ddot{x} = * \frac{16y}{a^4}$, &c. $\dot{x} = * * \frac{8y^2}{a^4}$, &c. $\ddot{x} = * * * \frac{8y^3}{3a^4}$, &c. $\dot{x} = * * * * \frac{2y^4}{3a^4}$, &c. and $x = * * * * * \frac{2y^5}{15a^4}$, &c. Again, $\dot{x}^6 = \frac{20\ddot{x}\dot{x} + 10\ddot{x}\dot{x} + 2\ddot{x}\dot{x}}{a^4}$, and again, $\dot{x}^7 = \frac{20\ddot{x}^2 + 10\ddot{x}\dot{x} + 12\ddot{x}\dot{x} + 2\ddot{x}\dot{x}}{a^4}$.

Here for $\dot{x}$, $\ddot{x}$, and $\ddot{x}$ writing 1, &c. $\frac{2}{a^2}$, &c. and $\frac{16}{a^4}$, &c. re-

spectively, 'tis $\dot{x}^7 = \frac{80 + 12 \times 16}{a^6}$, &c. $= \frac{272}{a^6}$, &c. Then $\dot{x}^6 = * \frac{272y}{a^6}$, &c. $\ddot{x} = * * \frac{136y^2}{a^6}$, &c. $\dot{x}^4 = * * * \frac{136y^3}{3a^6}$, &c. $\ddot{x} = * * * * \frac{34y^4}{3a^6}$, &c. $\ddot{x} = * * * * * \frac{34y^5}{15a^6}$, &c. $\dot{x} = * * * * * \frac{17y^6}{45a^6}$, &c. and $x = * * * * * \frac{17y^7}{315a^6}$, &c. That is, $x = y + \frac{y^3}{3a^2} + \frac{2y^5}{15a^4} + \frac{17y^7}{315a^6}$, &c.

For another Example, let us take the Equation $a^2y^2 - x^2y^2 = a^2$, or $y^2 = \frac{a^2}{a^2 - x^2} = 1$, &c. or $y = 1$, &c. and therefore $y = * x$, &c. Taking the Fluxions we shall have $2a^2\dot{y}y - 2x\dot{y}^2 - 2x^2\dot{y}\dot{y} = 0$, or $a^2\dot{y} - x\dot{y} - x^2\dot{y} = 0$, or $\dot{y} = \frac{x\dot{y}}{a^2 - x^2} = 0$, &c. And taking the Fluxions

again, 'tis $a^2\ddot{y} - \dot{y} - 3x\dot{y} - x^2\ddot{y} = 0$, or $\ddot{y} = \frac{\dot{y} + 3x\dot{y}}{a^2 - x^2} = \frac{1}{a^2}$, &c.

Therefore $\ddot{y} = * \frac{x}{a^2}$, &c. $\dot{y} = * * \frac{x^2}{2a^2}$, &c. and $y = * * * \frac{x^3}{6a^2}$, &c. Then $a^2\ddot{y} - 4\dot{y} - 5x\dot{y} - x^2\ddot{y} = 0$, and again $a^2\ddot{y} - 9\dot{y} - 7x\dot{y} - x^2\ddot{y} = 0$, or $\ddot{y} = \frac{9\dot{y} + 7x\dot{y}}{a^2 - x^2} = \frac{9\dot{y}}{a^2}$, &c. $= \frac{9}{a^4}$, &c. There-

fore $\ddot{y} = * \frac{9x}{a^4}$, &c. $\ddot{y} = * * \frac{9x^2}{2a^4}$, &c. $\ddot{y} = * * * \frac{3x^3}{2a^4}$, &c. $\dot{y} =$

* * * * $\frac{3x^4}{8a^4}$, &c. and $y =$ * * * * $\frac{3x^5}{40a^4}$, &c. Taking the Fluxions again, 'tis $a^2\dot{y} - 16\ddot{y} - 9x\dot{y} - x^2\ddot{y} = 0$, and again, $a^2\dot{y} - 25\dot{y} - 11x\dot{y} - x^2\ddot{y} = 0$, or $\dot{y} = \frac{25\dot{y} + 11x\dot{y}}{a^2 - x^2} = \frac{25}{a^2}\dot{y}$, &c. $= \frac{25 \times 9}{a^6}$, &c. Therefore $\dot{y} = \frac{25 \times 9}{a^6}x$, &c. $\ddot{y} = \frac{25 \times 9}{2a^6}x^2$, &c. $\ddot{\dot{y}} = \frac{25 \times 3}{2a^6}x^3$, &c. $\ddot{\ddot{y}} = \frac{25 \times 3}{8a^6}x^4$, &c. $\ddot{\ddot{\dot{y}}} = \frac{5 \times 3}{8a^6}x^5$, &c. $\dot{y} = \frac{5x^6}{16a^6}$, &c. and $y = \frac{5x^7}{112a^6}$, &c. Or $y = x + \frac{x^3}{6a^2} + \frac{3x^5}{40a^4} + \frac{5x^7}{112a^6}$, &c.

If we were required to extract the Root x out of the same Equation, $a^2y^2 - x^2y^2 - a^2\dot{x}^2 = 0$, (or to express the Sine by the Arch,) put $y = 1$, then $a^2 - x^2 - a^2\dot{x}^2 = 0$; or $\dot{x}^2 = 1 - \frac{x^2}{a^2}$, and therefore $\dot{x} = 1$, &c. and $x = y$, &c. Taking the

Fluxions 'tis $2\dot{x}x - 2a^2\dot{x}\ddot{x} = 0$, or $\ddot{x} = -\frac{\dot{x}}{a^2} = 0$, &c. Therefore $\dot{x} = 0$, &c. $x = 0$, &c. Taking the Fluxions

again, 'tis $\ddot{x} = -\frac{\dot{x}}{a^2} = -\frac{1}{a^2}$, &c. Thence $\ddot{x} = -\frac{y}{a^2}$, &c.

$\ddot{\dot{x}} = -\frac{y^2}{2a^2}$, &c. and $\ddot{x} = -\frac{y^3}{6a^2}$, &c. Again, $\ddot{\ddot{x}} = -\frac{\ddot{x}}{a^2}$,

and $\ddot{\ddot{x}} = -\frac{\ddot{x}}{a^2} = +\frac{1}{a^4}$, &c. Therefore $\ddot{\ddot{x}} = \frac{y}{a^4}$, &c. $\ddot{\ddot{\dot{x}}} = \frac{y^2}{2a^4}$, &c. $\ddot{\ddot{\ddot{x}}} = \frac{y^3}{6a^4}$, &c. $\ddot{\ddot{\ddot{\dot{x}}}} = \frac{y^4}{24a^4}$, &c. and $\ddot{\ddot{\ddot{x}}} = \frac{y^5}{120a^4}$, &c. Again, $\ddot{\ddot{\ddot{x}}} = -\frac{\ddot{\ddot{x}}}{a^2}$, and $\ddot{\ddot{\ddot{\dot{x}}}} = -\frac{\ddot{\ddot{\dot{x}}}}{a^2}$

$= -\frac{1}{a^6}$, &c. Therefore $\ddot{\ddot{\ddot{x}}} = \frac{y}{a^6}$, $\ddot{\ddot{\ddot{\dot{x}}}} = \frac{y^2}{2a^6}$,

&c. $\ddot{\ddot{\ddot{\ddot{x}}}} = \frac{y^3}{6a^6}$, &c. $\ddot{\ddot{\ddot{\ddot{\dot{x}}}}} = \frac{y^4}{24a^6}$, &c. $\ddot{\ddot{\ddot{\ddot{\ddot{x}}}}} = \frac{y^5}{120a^6}$, &c. $\ddot{\ddot{\ddot{\ddot{\ddot{\dot{x}}}}} = \frac{y^6}{720a^6}$, &c. and $\ddot{\ddot{\ddot{\ddot{\ddot{x}}}}} = \frac{y^7}{5040a^6}$, &c.

And therefore $x = y - \frac{y^3}{6a^2} + \frac{y^5}{120a^4} - \frac{y^7}{5040a^6}$, &c.

If it were required to extract the Root y out of this Equation, $a^2y^2 - x^2y^2 + m^2y^2 - m^2a^2 = 0$, (where $\dot{x} = 1$,) we might proceed.

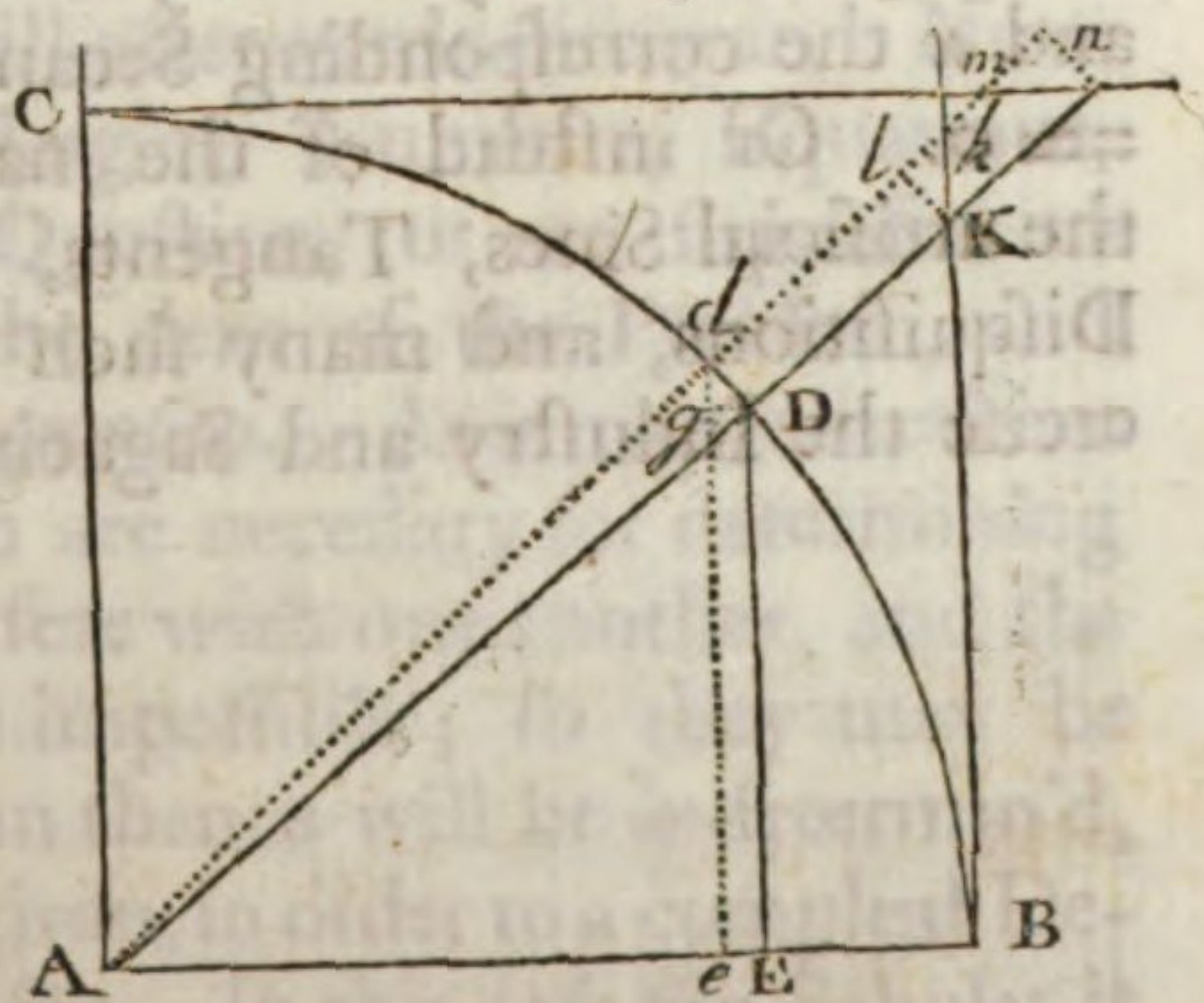
ceed thus. Because $\dot{y}^2 = \frac{m^2 a^2 - m^2 y^2}{a^2 - x^2} = m^2$, &c. 'tis $\dot{y} = m$, &c. and $y = * mx$, &c. Taking the Fluxions, we shall have $2a^2 \ddot{y} - 2x\dot{y}^2 - 2x^2 \ddot{y} + 2m^2 \dot{y}y = 0$, or $a^2 \ddot{y} - x\dot{y} - x^2 \ddot{y} + m^2 y = 0$, or $\ddot{y} = \frac{x\dot{y} - m^2 y}{a^2 - x^2} = 0$, &c. Therefore taking the Fluxions again, 'tis $a^2 \ddot{\ddot{y}} - \dot{y} - 3x\ddot{\ddot{y}} - x^2 \ddot{\ddot{y}} + m^2 \dot{y} = 0$, that is, $a^2 \ddot{\ddot{y}} + \overline{m^2 - 1} \times \dot{y} - 3x\ddot{\ddot{y}} - x^2 \ddot{\ddot{y}} = 0$, or $\ddot{\ddot{y}} = \frac{1 - m^2 \times \dot{y} + 3x\ddot{y}}{a^2 - x^2}$; and making $\dot{y} = m$, &c. 'tis $\ddot{\ddot{y}} = \frac{m \times 1 - m^2}{a^2}$, &c. and therefore $\ddot{y} = * \frac{m \times 1 - m^2}{a^2} x$, &c. $\dot{y} = * * \frac{m \times 1 - m^2}{2a^2} x^2$, &c. and $y = * * * \frac{m \times 1 - m^2}{2 \times 3a^2} x^3$, &c. Taking the Fluxions again, 'tis $a^2 \ddot{\ddot{\ddot{y}}} + \overline{m^2 - 4} \times \ddot{\ddot{y}} - 5x\ddot{\ddot{\ddot{y}}} - x^2 \ddot{\ddot{\ddot{y}}} = 0$; and again, $a^2 \ddot{\ddot{\ddot{\ddot{y}}}} + \overline{m^2 - 9} \times \ddot{\ddot{\ddot{y}}} - 7x\ddot{\ddot{\ddot{\ddot{y}}}} - x^2 \ddot{\ddot{\ddot{\ddot{y}}}} = 0$, or $\ddot{\ddot{\ddot{\ddot{y}}}} = \frac{9 - m^2 \times \ddot{\ddot{y}} + 7x\ddot{\ddot{\ddot{y}}}}{a^2 - x^2} = \frac{m \times 1 - m^2 \times 9 - m^2}{a^4}$, &c. Therefore $\ddot{\ddot{\ddot{y}}} = * \frac{m \times 1 - m^2 \times 9 - m^2}{a^4} x$, &c. $\ddot{\ddot{y}} = * * \frac{m \times 1 - m^2 \times 9 - m^2}{2a^4} x^2$, &c. $\ddot{y} = * * * \frac{m \times 1 - m^2 \times 9 - m^2}{2 \times 3a^4} x^3$, &c. $\dot{y} = * * * * \frac{m \times 1 - m^2 \times 9 - m^2}{2 \times 3 \times 4a^4} x^4$, &c. and $y = * * * * * \frac{m \times 1 - m^2 \times 9 - m^2}{2 \times 3 \times 4 \times 5a^4} x^5$, &c. And so on. Therefore we shall have $y = mx + m \times \frac{1 - m^2}{2 \times 3a^2} x^3 + m \times \frac{1 - m^2}{2 \times 3} \times \frac{9 - m^2}{4 \times 5a^4} x^5 + m \times \frac{1 - m^2}{2 \times 3} \times \frac{9 - m^2}{4 \times 5} \times \frac{25 - m^2}{6 \times 7a^6} x^7$, &c.

This Series is equivalent to a Theorem of our Author's, which (in another place) he gives us for Angular Sections. For if x be the Sine of any given Arch, to Radius a ; then will y be the Sine of another Arch, which is to the first Arch in the given Ratio of m to 1. Here if m be any odd Number, the Series will become finite; and in other cases it will be a converging Series.

And these Examples may be sufficient to explain this Method of Extraction of Roots; which, tho' it carries its own Demonstration along with it, yet for greater evidence may be thus farther illustrated. In Equations whose Roots (for example) may be represented by the general Series $y = A + Bx + Cx^2 + Dx^3$, &c. (which by due Reduction may be all Equations whatever,) the first Term A of the Root will be a given quantity, or perhaps $= 0$, which is to be known from the circumstances of the Question, or from the given Equation,

Equation, by Methods that have been abundantly explain'd already. Then making $x = 1$, we shall have $y = B + 2Cx + 3Dx^2$, &c. where B likewise is a constant quantity, or perhaps $= 0$, and represents the first Term of the Series y . This therefore is to be derived from the first Fluxional Equation, either given or else to be found; and then, because it is $\dot{y} = B$, &c. by taking the Fluents it will be $y = * Bx$, &c. whence the second Term of the Root will be known. Then because it is $\ddot{y} = 2C + 6Dx$, &c. or because the constant quantity $2C$ will represent the first Term of $\ddot{y}$; this is to be derived from the second Fluxional Equation, either given or to be found. And then, because it is $\ddot{y} = 2C$, &c. by taking the Fluents it will be $\dot{y} = * 2Cx$, &c. and again $y = * * Cx^2$, &c. by which the third Term of the Root will be known. Then because it is $\ddot{\dot{y}} = 6D$, &c. or because the constant quantity $6D$ will represent the first Term of the Series $\ddot{\dot{y}}$; this is to be derived from the third Fluxional Equation. And then, because it is $\ddot{\dot{y}} = 6D$, &c. by taking the Fluents it will be $\ddot{y} = * 6Dx$, &c. $\dot{y} = * * 3Dx^2$, &c. and $y = * * * Dx^3$, &c. by which the fourth Term of the Root will be known. And so for all the subsequent Terms. And hence it will not be difficult to observe the composition of the Coefficients in most cases, and thereby discover the Law of Continuation, in such Series as are notable and of general use.

If you should desire to know how the foregoing Trigonometrical Equations are derived from the Circle, it may be shewn thus: on the Center A , with Radius $AB = a$, let the Quadrantal Arch BC be described, and draw the Radius AC . Draw the Tangent BK , and through any point of the Circumference D , draw the Secant ADK , meeting the Tangent in K . At any other point d of the Circumference, but as near to D as may be, draw the Secant Adk , meeting BK in k ; on Center A , with Radius AK , describe the Arch Kl , meeting Ak in l . Then supposing the point d continually to approach towards D , till it finally coincides with it, the Trilineum Klk will continually approach to a right-lined Triangle, and to similitude with the Triangle ABK : So that when Dd is a Moment

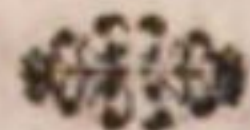


Moment of the Circumference, it will be $\frac{Kk}{Dd} = \frac{Kk}{Kl} \times \frac{Kl}{Dd} = \frac{AK}{AB} \times \frac{AK}{AB}$. Make $AB = a$, the Tangent $BK = x$, and the Arch $BD = y$; and instead of the Moments Kk and Dd , substitute the proportional Fluxions $\dot{x}$ and $\dot{y}$, and it will be $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{a^2}$, or $a^2\dot{y} + x^2\dot{y} - a^2\dot{x} = 0$.

From D to AB and de let fall the Perpendiculars DE and Dg , which Dg meets de , parallel to DE , in g . Then the ultimate form of the Trilineum Ddg will be that of a right-lined Triangle similar to DAE . Whence $Dd : dg :: AD : AE = \sqrt{ADg - DEg}$. Make $AD = a$, $BD = y$, and $DE = x$; and for the Moments Dd , dg , substitute their proportional Fluxions $\dot{y}$ and $\dot{x}$, and it will be $\dot{y} : \dot{x} :: a : \sqrt{a^2 - x^2}$. Or $\dot{y}^2 : \dot{x}^2 :: a^2 : a^2 - x^2$, or $a^2\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$.

Hence the Fluxion of an Arch, whose right Sine is x , being express'd by $\frac{a\dot{x}}{\sqrt{a^2 - x^2}}$; and likewise the Fluxion of an Arch, whose right Sine is y , being express'd by $\frac{a\dot{y}}{\sqrt{a^2 - y^2}}$; if these Arches are to each other as 1 to m , their Fluxions will be in the same proportion, and *vice versa*. Therefore $\frac{a\dot{x}}{\sqrt{a^2 - x^2}} : \frac{a\dot{y}}{\sqrt{a^2 - y^2}} :: 1 : m$, or $\frac{m\dot{x}}{\sqrt{a^2 - x^2}} = \frac{\dot{y}}{\sqrt{a^2 - y^2}}$, or $\frac{m^2\dot{x}^2}{a^2 - x^2} = \frac{\dot{y}^2}{a^2 - y^2}$, or putting $\dot{x} = 1$, 'tis $a^2\dot{y}^2 - x^2\dot{y}^2 - m^2a^2 + m^2y^2 = 0$; the same Equation as before resolved.

We might derive other Fluxional Equations, of a like nature with these, which would be accommodated to Trigonometrical uses. As if y were the Circular Arch, and x its versed Sine, we should have the Equation $2ax\dot{y}^2 - x^2\dot{y}^2 - a^2\dot{x}^2 = 0$. Or if y were the Arch, and x the corresponding Secant, it would be $x^4\dot{y}^2 - a^2x^2\dot{y}^2 - a^4\dot{x}^2 = 0$. Or instead of the natural, we might derive Equations for the artificial Sines, Tangents, Secants, &c. But I shall leave these Disquisitions, and many such others that might be proposed, to exercise the Industry and Sagacity of the Learner.



SECT. VI. *An Analytical Appendix, explaining some Terms and Expressions in the foregoing work.*

BECAUSE mention has been frequently made of *given Equations*, and others *assumed ad libitum*, and the like; I shall take occasion from hence, by way of Appendix, to attempt some kind of explanation of this Mathematical Language, or of the Terms *given, assign'd, assumed, and required* Quantities or Equations, which may give light to some things that may otherwise seem obscure, and may remove some doubts and scruples, which are apt to arise in the Mind of a Learner. Now the origin of such kind of Expressions in all probability seems to be this. The whole affair of pursuing Mathematical Inquiries, or of resolving Problems, is supposed (tho' tacitely) to be transacted between two Persons, or Parties, the Proposer and the Resolver of the Problem, or (if you please) between the Master (or Instructor) and his Scholar. Hence this, and such like Phrases, *datam rectam, vel datum angulum, in imperatâ ratione secare*. As Examples instruct better than Precepts, or perhaps when both are join'd together they instruct best, the Master is suppos'd to propose a Question or Problem to his Scholar, and to chuse such Terms and Conditions as he thinks fit; and the Scholar is obliged to solve the Problem with those limitations and restrictions, with those Terms and Conditions, and no other. Indeed it is required on the part of the Master, that the Conditions he proposes may be consistent with one another; for if they involve any inconsistency or contradiction, the Problem will be unfair, or will become absurd and impossible, as the Solution will afterwards discover. Now these Conditions, these Points, Lines, Angles, Numbers, Equations, &c. that at first enter the state of the Question, or are supposed to be chosen or given by the Master, are the *data* of the Problem, and the Answers he expects to receive are the *quæsitæ*. As it may sometimes happen, that the *data* may be more than are necessary for determining the Question, and so perhaps may interfere with one another, and the Problem (as now proposed) may become impossible; so they may be fewer than are necessary, and the Problem thence will be indetermin'd, and may require other Conditions to be given, in order to a compleat Determination, or perfectly to fulfil the *quæsitæ*. In this case the Scholar is to supply what is wanting, and at his discretion may *assume* such and so many other Terms and Conditions, Equations and Limitations, as he finds

will be necessary to his purpose, and will best conduce to the simplest, the easiest, and neatest Solution that may be had, and yet in the most general manner. For it is convenient the Problem should be proposed as particular as may be, the better to fix the Imagination; and yet the Solution should be made as general as possible, that it may be the more instructive, and extend to all cases of a like nature.

Indeed the word *datum* is often used in a sense which is something different from this, but which ultimately centers in it. As that is call'd a *datum*, when one quantity is not immediately given, but however is necessarily infer'd from another, which other perhaps is necessarily infer'd from a third, and so on in a continued Series, till it is necessarily infer'd from a quantity, which is known or given in the sense before explain'd. This is the Notion of *Euclid's data*, and other Analytical Argumentations of that kind. Again, that is often call'd a *given quantity*, which always remains constant and invariable, while other quantities or circumstances vary; because such as these only can be the given quantities in a Problem, when taken in the foregoing sense.

To make all this the more sensible and intelligible, I shall have recourse to a few practical instances, by way of Dialogue, (which was the old didactic method,) between Master and Scholar; and this only in the common Algebra or Analyticks, in which I shall borrow my Examples from our Author's admirable Treatise of Universal Arithmetick. The chief artifice of this manner of Solution will consist in this, that as fast as the Master proposes the Conditions of his Question, the Scholar applies those Conditions to use, argues from them Analytically, makes all the necessary deductions, and derives such consequences from them, in the same order they are proposed, as he apprehends will be most subservient to the Solution. And he that can do this in all cases, after the surest, simplest, and readiest manner, will be the best *ex-tempore* Mathematician. But this method will be best explain'd from the following Examples.

I. M. *A Gentleman being willing to distribute Alms - - -* S. Let the Sum he intended to distribute be represented by x . M. *Among some poor people.* S. Let the number of poor be y , then $\frac{x}{y}$ would have been the share of each. M. *He wanted 3 shillings - - -* S. Make $3 = a$, for the sake of universality, and let the pecuniary Unit be one Shilling; then the Sum to be distributed would have been $x + a$,
and

and the share of each would have been $\frac{x+a}{y}$. M. So that each might receive 5 shillings. S. Make $5 = b$, then $\frac{x+a}{y} = b$, whence $x = by - a$. M. Therefore he gave every one 4 shillings. S. Make $4 = c$, then the Money distributed will be cy . M. And he has 10 shillings remaining. S. Make $10 = d$, then $cy + d$ was the Money he intended at first to distribute; or $cy + d = (x =) by - a$, or $y = \frac{a+d}{b-c}$. M. What was the number of poor people? S. The number was $y = \frac{a+d}{b-c} = \frac{3+10}{5-4} = 13$. M. And how much Alms did he at first intend to distribute? S. He had at first $x = by - a = 5 \times 13 - 3 = 62$ shillings. M. How do you prove your Solution? S. His Money was at first 62 shillings, and the number of poor people was 13. But if his Money had been $62 + 3 = 65 = 13 \times 5$ shillings, then each poor person might have received 5 shillings. But as he gives to each 4 shillings, that will be $13 \times 4 = 52$ shillings distributed in all, which will leave him a Remainder of $62 - 52 = 10$ shillings.

II. M. A young Merchant, at his first entrance upon business, began the World with a certain Sum of Money. S. Let that Sum be x , the pecuniary Unit being one Pound. M. Out of which, to maintain himself the first year, he expended 100 pounds. S. Make the given number $100 = a$; then he had to trade with $x - a$. M. He traded with the rest, and at the end of the year had improved it by a third part. S. For universality-sake I will assume the general number n , and will make $\frac{1}{3} = n - 1$, (or $n = \frac{4}{3}$;) then the Improvement was $n - 1 \times x - a = nx - na - x + a$, and the Trading-stock and Improvement together, at the end of the first year, was $nx - na$. M. He did the same thing the second year. S. That is, his whole Stock being now $nx - na$, deducting a , his Expences for this year, he would have $nx - na - a$ for a Trading-stock, and $n - 1 \times nx - na - a$, or $n^2x - n^2a - nx + a$ for this year's Improvement, which together make $n^2x - n^2a - na$ for his Estate at the end of the second year. M. As also the third year. S. His whole Stock being now $n^2x - n^2a - na$, taking out his Expences for the third year, his Trading-stock will be $n^2x - n^2a - na - a$, and the Improvement this year will be $n - 1 \times n^2x - n^2a - na - a$, or $n^3x - n^3a - n^2x + a$, and the Stock and Improvement together, or his whole Estate at the end of the third year will be $n^3x - n^3a - n^2a - na$, or in a better form $n^3x + \frac{n^3-1}{1-n}na$. In like manner

if he proceeded thus the fourth year, his Estate being now $n^3x - n^3a - n^2a - na$, taking out this year's Expence, his Trading-stock will be $n^3x - n^3a - n^2a - na - a$, and this year's Improvement is $n - 1 \times n^3x - n^3a - n^2a - na - a$, or $n^4x - n^4a - n^3x + a$, which added to his Trading-stock will be $n^4x - n^4a - n^3a - n^2a - na$, or $n^4x + \frac{n^4-1}{1-n}na$, for his Estate at the end of the fourth

year. And so, by Induction, his Estate will be found $n^5x + \frac{n^5-1}{1-n}na$ at the end of the fifth year. And universally, if I assume the general Number m , his Estate will be $n^mx + \frac{n^m-1}{1-n}na$ at the end of any number of years denoted by m . M. But he made his Estate double to what it was at first. S. Make $2 = b$, then $n^mx + \frac{n^m-1}{1-n}na = bx$, or $x = \frac{n^m-1}{n-1 \times n^m-b}na$. M. At the end of 3 years.

S. Then $m = 3$, $a = 100$, $b = 2$, $n = \frac{4}{3}$, and therefore $x = \frac{\frac{4^3}{3^3} - 1}{\frac{4}{3} \times \frac{4^3}{3^3} - 2} \times \frac{4}{3} \times 100 = \frac{4^3-3^3}{4^3-2 \times 3^3} \times 400 = \frac{64-27}{64-54} \times 400 = \frac{37}{10} \times$

$400 = 1480$. M. What was his Estate at first? S. It was 1480 pounds.

III. M. Two Bodies A and B are at a given distance from each other. S. As their distance is said to be given, though it is not so actually, I may therefore assume it. Let the initial distance of the Bodies be $59 = e$, and let the Linear Unit be one Mile. M. And move equably towards one another. S. Let x represent the whole space described by A before they meet; then will $e - x$ be the whole space described by B. M. With given Velocities. S. I will assume the Velocity of A to be such, that it will move $7 = c$ Miles in $2 = f$ Hours, the Unit of Time being one Hour. Then because it is $c : f :: x : \frac{fx}{c}$, A will move his whole space x in the time $\frac{fx}{c}$. Also I will assume the Velocity of B to be such, that it will move $8 = d$ Miles in $3 = g$ Hours. Then because it is $d : g :: e - x : \frac{e-x}{d}g$, B will move his whole space $e - x$ in the time $\frac{e-x}{d}g$. M. But A moves a given time - - - S. Let that time be $1 = b$ Hour. M. Before B begins to move. S. Then A's time is equal to B's time added to the time b , or $\frac{fx}{c} = \frac{e-x}{d}g + b$.

M.

M. Where will they meet, or what will be the space that each will have described? S. From this Equation we shall have $x = \frac{eg + db}{df + cg}c$

$= \frac{59 \times 3 + 8 \times 1}{8 \times 2 + 7 \times 3} \times 7 = \frac{185}{37} \times 7 = 5 \times 7 = 35$ Miles, which will be the whole space described by A. Then $e - x = 59 - 35 = 24$ Miles will be the whole space described by B.

IV. M. If 12 Oxen can be maintained by the Pasture of $3\frac{1}{3}$ Acres of Meadow-ground for 4 weeks, S. Make $12 = a$, $3\frac{1}{3} = b$, $4 = c$; then assuming the general Numbers e , f , h , to be determin'd afterwards as occasion shall require, we shall have by analogy

	Oxen	Pasture	Time
If	a	b	c
Then	ae	be	c
Also	$\frac{ae}{b}$	e	c
And	$\frac{ace}{b}$	e	1
Also	$\frac{ace}{bf}$	e	f
Also	$\frac{ace}{bb}$	e	b

require during

M. And if, because of the continual growth of the Grass after the four weeks, it be found that 21 Oxen can be maintain'd by the pasture of 10 such Acres for 9 weeks, S. Make $21 = d$, $e = 10$, $f = 9$; then because on this supposition, the Oxen d require the pasture e during the time f ; and in the former case the Oxen $\frac{ace}{bf}$ required the same pasture during the same time: Therefore the growth of the Grass of the quantity of pasture e , (commencing after 4 or c weeks, and continuing to the end of the Time f , or during the whole time $f - c$;) is such, as alone was sufficient to maintain the difference of the Oxen, or the number $d - \frac{ace}{bf}$, for the whole time f . Then reciprocally that growth would be sufficient to maintain the number of Oxen $df - \frac{ace}{b}$ for the time 1, or the number of Oxen $\frac{df}{b} - \frac{ace}{bb}$ for the time b . And because this growth will be proportional to the time, and will maintain a greater number of Oxen in proportion as the time is greater; we shall have

Time

$$\begin{array}{ccccc} \text{Time} & \text{Oxen} & & \text{Time} & \text{Oxen} \\ f - c & \frac{df}{b} - \frac{ace}{bh} :: b - c & . & \frac{b-c}{f-c} \text{ into } \frac{df}{b} - \frac{ace}{bh}, \end{array}$$

which will be the number of Oxen that may be maintain'd by the growth only of the pasture e , during the whole time b . But it was found before, that without this growth of the Grass, the Oxen $\frac{ace}{bh}$ might be maintain'd by the pasture e for the time b . Therefore these two together, or $\frac{ace}{bh} + \frac{b-c}{f-c} \times \frac{bdf - ace}{bh}$, will be the number of Oxen that may be maintain'd by the pasture e , and its growth together, during the time b . M. *How many Oxen may be maintain'd by 24 Acres of such pasture for 18 weeks?* S. Suppose x to be that number of Oxen, and make $24 = g$, and $b = 18$. Then by analogy

$$\begin{array}{ccc} & \text{Oxen} & \text{Pasture} \\ \text{If} & \left\{ \begin{array}{c} x \\ ex \\ \frac{ex}{g} \end{array} \right\} & \text{require } \left\{ \begin{array}{c} g \\ eg \\ e \end{array} \right\} \text{ during the time } b. \\ \text{Then} & & \\ \text{And} & & \end{array}$$

$$\begin{aligned} \text{And consequently } \frac{ex}{g} &= \frac{ace}{bh} + \frac{b-c}{f-c} \times \frac{bdf - ace}{bh}, \text{ or } x = \frac{acg}{bh} + \frac{b-c}{f-c} \\ &\times \frac{dfg}{eb} - \frac{acg}{bh} = \frac{ac}{b} + \frac{b-c}{f-c} \times \frac{df}{e} - \frac{ac}{b} \text{ into } \frac{g}{b} = \frac{12 \times 4}{3\frac{1}{3}} + \frac{18-4}{9-4} \times \\ &\frac{21 \times 9}{10} - \frac{12 \times 4}{3\frac{1}{3}} \text{ into } \frac{24}{18} = 36. \end{aligned}$$

V. M. *If I have an Annuity - - -* S. Let x be the present value of 1 pound to be received 1 year hence, then (by analogy) x^2 will be the present value of 1 pound to be received 2 years hence, &c. and in general, x^m will be the present value of 1 pound to be received m years hence. Therefore, in the case of an Annuity, the Series $x + x^2 + x^3 + x^4$, &c. to be continued to so many Terms as there are Units in m , will be the present value of the whole Annuity of 1 pound, to be continued for m years. But because $\frac{x - x^{m+1}}{1 - x} = x + x^2 + x^3 + x^4$, &c. continued to so many Terms as there are Units in m , (as may appear by Division;) therefore $\frac{x - x^{m+1}}{1 - x}$ will represent the Amount of an Annuity of 1 pound, to be continued for m years. M. Of Pounds. S. Make $= a$,

$\equiv a$, then the Amount of this Annuity for m years will be $\frac{x-x^{m+1}}{1-x} a$. M. To be continued for 5 years successively. S. Then $m=5$. M. Which I sell for pounds in ready Money. S. Make

$$\equiv c, \text{ then } \frac{x-x^{m+1}}{1-x} a \equiv c, \text{ or } x^{m+1} - \frac{c}{a} + 1 \times x + \frac{c}{a} \equiv 0.$$

In any particular case the value of x may be found by the Resolution of this affected Equation. M. What Interest am I allow'd per centum per annum? S. Make $100 \equiv b$; then because x is the present value of 1 pound to be received 1 year hence, or (which is the same thing) because the present Money x , if put out to use, in 1 year will produce 1 pound; the Interest alone of 1 pound for 1 year will be $1-x$, and therefore the Interest of 100 (or b) pounds for 1 year will be $b-bx$, which will be known when x is known.

And this might be sufficient to shew the conveniency of this Method; but I shall farther illustrate it by one Geometrical Problem, which shall be our Author's LVII.

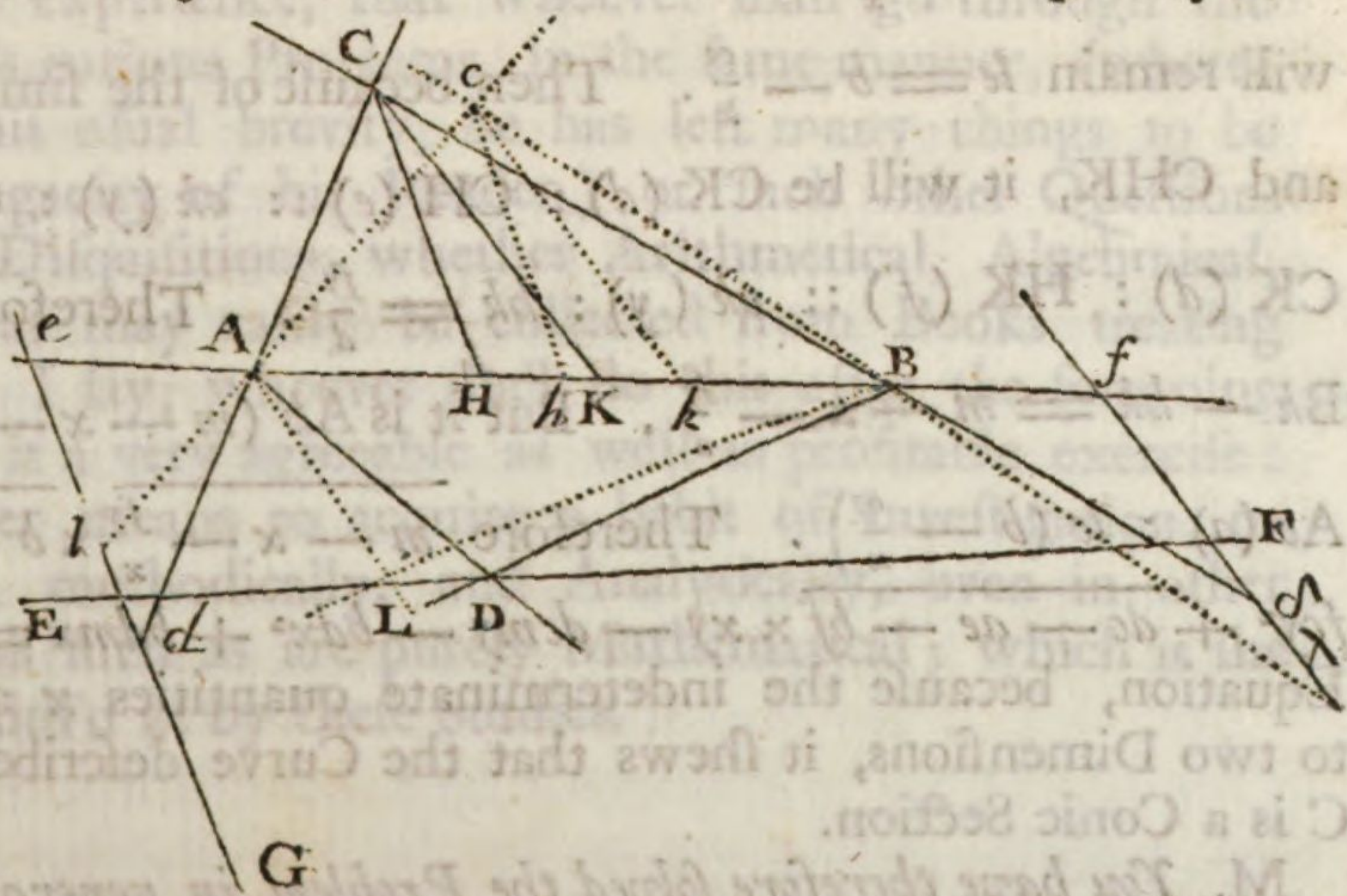
VI. M. In the right Line AB I give you the two points A and B. S. Then their distance $AB \equiv m$ is given also. M. As likewise the two points C and D out of the Line AB. S. Then consequently the figure ACBD is

given in magnitude and specie; and producing CA and CB towards d and δ , I can take $Ad \equiv AD$, and $B\delta \equiv BD$.

M. Also I give you the indefinite right Line EF in position, passing thro' the given point D.

S. Then the Angles ADE and BDF are given, to which (producing AB both ways, if need be, to e and f ;) I can make the Angles Ade and Bdf equal respectively, and that will determine the points e and f , or the Lines $Ae \equiv a$, and $Bf \equiv c$. And because de and δf are thereby known, I can continue de to G, so that $dG \equiv \delta f$, and make the given line $eG \equiv b$. Likewise I can draw CH

and



and CK parallel to ed and $f\delta$ respectively, meeting AB in H and K; and because the Triangle CHK will be given in magnitude and specie, I will make $CK = d$, $CH = e$, and $HK = f$. M. Now let the given Angles CAD and CBD be conceived to revolve about the given points or Poles A and B. S. Then the Lines AD and CAD will move into another situation AL and cAl, so as that the Angles DAL, dAl, and CAc will be equal. Also the Lines BD and CB δ will obtain a new situation BL and cB λ , so as that the Angles DBL, $\delta B\lambda$ and CBc will be equal. M. And let D, the Intersection of the Lines AD and BD, always move in the right Line EF. S. Then the new point of Intersection L is in EF; then the Triangles DAL and dAl, as also DBL and $\delta B\lambda$, are equal and similar; then $dI = DL = \delta\lambda$, and therefore $GI = f\lambda$. M. What will be the nature of the Curve described by the other point of Intersection C? S. From the new point of Intersection c to AB, I will draw the Lines cb and ck, parallel to CH and CK respectively. Then will the Triangle cbk be given in specie, though not in magnitude, for it will be similar to CHK. Also the Triangle Bck will be similar to B λf . And the indefinite Line Bk $= x$ may be assumed for an Absciss, and ck $= y$ may be the corresponding Ordinate to the Curve Cc. Then because it is Bk (x) : ck (y) :: Bf (c) : f $\lambda = \frac{y}{x} = GI$. Subtract this from Ge $= b$, and there will remain le $= b - \frac{cy}{x}$. Then because of the similar Triangles cbk and CHK, it will be CK (d) : CH (e) :: ck (y) : cb $= \frac{ey}{d}$. And CK (d) : HK (f) :: ck (y) : bk $= \frac{fy}{d}$. Therefore Ab $= AB - Bk - bk = m - x - \frac{fy}{d}$. But it is Ab $(m - x - \frac{fy}{d})$: cb $(\frac{ey}{d})$:: Ae (a) : le $(b - \frac{cy}{x})$. Therefore $\frac{m - x - \frac{fy}{d}}{x} \times b - \frac{cy}{d} = \frac{aey}{d}$, or $fcy^2 + dc - ae - bf \times xy - dcmy - bdx^2 + bdmx = 0$. In which Equation, because the indeterminate quantities x and y arise only to two Dimensions, it shews that the Curve described by the point C is a Conic Section.

M. You have therefore solved the Problem in general, but you should now apply your Solution to the several species of Conic Sections in particular. S. That may easily be done in the following manner: Make $\frac{ae + bf - cd}{c} = 2p$, and then the foregoing Equation will become $fcy^2 - 2pcxy - dcmy - bdx^2 + bdmx = 0$, and by extracting

tracting the Square-root it will be $y = \frac{p}{f}x + \frac{dm}{2f} \pm$

$$\sqrt{\frac{pp}{ff} + \frac{bd}{fc} \times x^2 + \frac{pdm}{ff} - \frac{bdm}{fc} \times x + \frac{d^2m^2}{4ff}}. \text{ Now here it is plain,}$$

that if the Term $\frac{pp}{ff} + \frac{bd}{fc} \times x^2$ were absent, or if $\frac{pp}{ff} + \frac{bd}{fc} = 0$, or

$\frac{pp}{ff} = -\frac{bd}{fc}$; that is, if the quantity $\frac{bd}{fc}$ (changing its sign) should

be equal to $\frac{pp}{ff}$, then the Curve would be a Parabola. But if the

same Term were present, and equal to some affirmative quantity,

that is, if $\frac{pp}{ff} + \frac{bd}{fc}$ be affirmative, (which will always be when

$\frac{bd}{fc}$ is affirmative, or if it be negative and less than $\frac{pp}{ff}$;) the Curve

will be an Hyperbola. Lastly, if the same Term were present and

negative, (which can only be when $\frac{bd}{fc}$ is negative, and greater than

$\frac{pp}{ff}$;) the Curve will be an Ellipsis or a Circle.

I should make an apology to the Reader, for this Digression

from the Method of Fluxions, if I did not hope it might contribute

to his entertainment at least, if not to his improvement. And I am

fully convinced by experience, that whoever shall go through the

rest of our Author's curious Problems, in the same manner, (where-

in, according to his usual brevity, he has left many things to be

supply'd by the sagacity of his Reader,) or such other Questions

and Mathematical Disquisitions, whether Arithmetical, Algebraical,

Geometrical, &c. as may easily be collected from Books treating

on these Subjects; I say, whoever shall do this after the foregoing

manner, will find it a very agreeable as well as profitable exercise:

As being the proper means to acquire a habit of Investigation, or

of arguing surely, methodically, and Analytically, even in other

Sciences as well as such as are purely Mathematical; which is the

great end to be aim'd at by these Studies.

SECT. VII. *The Conclusion; containing a short recapitulation or review of the whole.*

WE are now arrived at a period, which may properly enough be call'd *the conclusion of the Method of Fluxions and Infinite Series*; for the design of this Method is to teach the nature of Series in general, and of Fluxions and Fluents, what they are, how they are derived, and what Operations they may undergo; which design (I think) may now be said to be accomplish'd. As to the application of this Method, and the uses of these Operations, which is all that now remains, we shall find them insisted on at large by the Author in the curious Geometrical Problems that follow. For the whole that can be done, either by Series or by Fluxions, may easily be reduced to the Resolution of Equations, either Algebraical or Fluxional, as it has been already deliver'd, and will be farther apply'd and pursued in the sequel. I have continued my Annotations in a like manner upon that part of the Work, and intended to have added them here; but finding the matter to grow so fast under my hands, and seeing how impossible it was to do it justice within such narrow limits, and also perceiving this work was already grown to a competent size; I resolv'd to lay it before the Mathematical Reader unfinish'd as it is, reserving the completion of it to a future opportunity, if I shall find my present attempts to prove acceptable. Therefore all that remains to be done here is this, to make a kind of review of what has been hitherto deliver'd, and to give a summary account of it, in order to acquit myself of a Promise I made in the Preface. And having there done this already, as to the Author's part of the work, I shall now only make a short recapitulation of what is contain'd in my own Comment upon it.

And first in my Annotations upon what I call the Introduction, or the Resolution of Equations by infinite Series, I have amply pursued a useful hint given us by the Author, that Arithmetick and Algebra are but one and the same Science, and bear a strict analogy to each other, both in their Notation and Operations; the first computing after a definite and particular manner, the latter after a general and indefinite manner: So that both together compose but one uniform Science of Computation. For as in common Arithmetick we reckon by the Root *Ten*, and the several Powers of that Root; so in Algebra, or Analyticks, when the Terms are orderly dispos'd

dispos'd as is prescribed, we reckon by any other Root and its Powers, or we may take any general Number for the Root of our Arithmetical Scale, by which to express and compute any Numbers required. And as in common Arithmetick we approximate continually to the truth, by admitting Decimal Parts *in infinitum*, or by the use of Decimal Fractions, which are composed of the reciprocal Powers of the Root *Ten*; so in our Author's improved Algebra, or in the Method of infinite converging Series, we may continually approximate to the Number or Quantity required, by an orderly succession of Fractions, which are composed of the reciprocal Powers of any Root in general. And the known Operations in common Arithmetick, having a due regard to Analogy, will generally afford us proper patterns and specimens, for performing the like Operations in this Universal Arithmetick.

Hence I proceed to make some Inquiries into the nature and formation of infinite Series in general, and particularly into their two principal circumstances of Convergency and Divergency; wherein I attempt to shew, that in all such Series, whether converging or diverging, there is always a Supplement, which if not express'd is however to be understood; which Supplement, when it can be ascertained and admitted, will render the Series finite, perfect, and accurate. That in diverging Series this Supplement must indispensably be admitted and exhibited, or otherwise the Conclusion will be imperfect and erroneous. But in converging Series this Supplement may be neglected, because it continually diminishes with the Terms of the Series, and finally becomes less than any assignable quantity. And hence arises the benefit and conveniency of infinite converging Series; that whereas that Supplement is commonly so implicated and entangled with the Terms of the Series, as often to be impossible to be extricated and exhibited; in converging Series it may safely be neglected, and yet we shall continually approximate to the quantity required. And of this I produce a variety of Instances, in numerical and other Series.

I then go on to shew the Operations, by which infinite Series are either produced, or which, when produced, they may occasionally undergo. As first when simple specious Equations, or pure Powers, are to be resolved into such Series, whether by Division, or by Extraction of Roots; where I take notice of the use of the afore-mention'd Supplement, by which Series may be render'd finite, that is, may be compared with other quantities, which are consider'd as given. I then deduce several useful Theorems, or other Artifices,

for the more expeditious Multiplication, Division, Involution, and Evolution of infinite Series, by which they may be easily and readily managed in all cases. Then I shew the use of these in pure Equations, or Extractions; from whence I take occasion to introduce a new praxis of Resolution, which I believe will be found to be very easy, natural, and general, and which is afterwards apply'd to all species of Equations.

Then I go on with our Author to the *Exegetis numerosa*, or to the Solution of affected Equations in Numbers; where we shall find his Method to be the same that has been publish'd more than once in other of his pieces, to be very short, neat, and elegant, and was a great Improvement at the time of its first publication. This Method is here farther explain'd, and upon the same Principles a general Theorem is form'd, and distributed into several subordinate Cases, by which the Root of any Numerical Equation, whether pure or affected, may be computed with great exactness and facility.

From Numeral we pass on to the Resolution of Literal or Specious affected Equations by infinite Series; in which the first and chief difficulty to be overcome, consists in determining the forms of the several Series that will arise, and in finding their initial Approximations. These circumstances will depend upon such Powers of the Relate and Correlate Quantities, with their Coefficients, as may happen to be found promiscuously in the given Equation. Therefore the Terms of this Equation are to be disposed *in longum* & *in latum*, or at least the Indices of those Powers, according to a combined Arithmetical Progression *in plano*, as is there explain'd; or according to our Author's ingenious Artifice of the Parallelogram and Ruler, the reason and foundation of which are here fully laid open. This will determine all the cases of exterior Terms, together with the Progressions of the Indices; and therefore all the forms of the several Series that may be derived for the Root, as also their initial Coefficients, Terms, or Approximations.

We then farther prosecute the Resolution of Specious Equations, by diverse Methods of Analysis; or we give a great variety of Processes, by which the Series for the Roots are easily produced to any number of Terms required. These Processes are generally very simple, and depend chiefly upon the Theorems before deliver'd, for finding the Terms of any Power or Root of an infinite Series. And the whole is illustrated and exemplify'd by a great variety of Instances, which are chiefly those of our Author.

The Method of infinite Series being thus sufficiently discuss'd, we make a Transition to the Method of Fluxions, wherein the nature and foundation of that Method is explain'd at large. And some general Observations are made, chiefly from the Science of Rational Mechanicks, by which the whole Method is divided and distinguish'd into its two grand Branches or Problems, which are the Direct and Inverse Methods of Fluxions. And some preparatory Notations are deliver'd and explain'd, which equally concern both these Methods.

I then proceed with my Annotations upon the Author's first Problem, or the Relation of the flowing Quantities being given, to determine the Relation of their Fluxions. I treat here concerning Fluxions of the first order, and the method of deducing their Equations in all cases. I explain our Author's way of taking the Fluxions of any given Equation, which is much more general and scientifick than that which is usually follow'd, and extends to all the varieties of Solutions. This is also apply'd to Equations involving several flowing Quantities, by which means it likewise comprehends those cases, in which either compound, irrational, or mechanical Quantities may be included. But the Demonstration of Fluxions, and of the Method of taking them, is the chief thing to be consider'd here; which I have endeavour'd to make as clear, explicite, and satisfactory as I was able, and to remove the difficulties and objections that have been rais'd against it: But with what success I must leave to the judgment of others.

I then treat concerning Fluxions of superior orders, and give the Method of deriving their Equations, with its Demonstration. For tho' our Author, in this Treatise, does not expressly mention these orders of Fluxions, yet he has sometimes recourse to them, tho' tacitely and indirectly. I have here shewn, that they are a necessary result from the nature and notion of first Fluxions; and that all these several orders differ from each other, not absolutely and essentially, but only relatively and by way of comparison. And this I prove as well from Geometry as from Analyticks; and I actually exhibit and make sensible these several orders of Fluxions.

But more especially in what I call the Geometrical and Mechanical Elements of Fluxions, I lay open a general Method, by the help of Curve-lines and their Tangents, to represent and exhibit Fluxions and Fluents in all cases, with all their concomitant Symptoms and Affections,

Affections, after a plain and familiar manner, and that even to ocular view and inspection. And thus I make them the Objects of Sense, by which not only their existence is proved beyond all possible contradiction, but also the Method of deriving them is at the same time fully evinced, verified, and illustrated.

Then follow my Annotations upon our Author's second Problem, or the Relation of the Fluxions being given, to determine the Relation of the flowing Quantities or Fluents; which is the same thing as the Inverse Method of Fluxions. And first I explain (what our Author calls) a particular Solution of this Problem, because it cannot be generally apply'd, but takes place only in such Fluxional Equations as have been, or at least might have been, previously derived from some finite Algebraical or Fluential Equations. Whereas the Fluxional Equations that usually occur, and whose Fluents or Roots are required, are commonly such as, by reason of Terms either redundant or deficient, cannot be resolved by this particular Solution; but must be refer'd to the following general Solution, which is here distributed into these three Cases of Equations.

The first Case of Equations is, when the Ratio of the Fluxions of the Relate and Correlate Quantities, (which Terms are here explain'd,) can be express'd by the Terms of the Correlate Quantity alone; in which Case the Root will be obtain'd by an easy process: In finite Terms, when it may be done, or at least by an infinite Series. And here a useful Rule is explain'd, by which an infinite Expression may be always avoided in the Conclusion, which otherwise would often occur, and render the Solution inexplicable.

The second Case of Equations comprehends such Fluxional Equations, wherein the Powers of the Relate and Correlate Quantities, with their Fluxions, are any how involved. Tho' this Case is much more operose than the former, yet it is solved by a variety of easy and simple Analyses, (more simple and expeditious, I think, than those of our Author,) and is illustrated by a numerous collection of Examples.

The third and last Case of Fluxional Equations is, when there are more than two Fluents and their Fluxions involved; which Case, without much trouble, is reduced to the two former. But here are also explain'd some other matters, farther to illustrate this Doctrine; as the Author's Demonstration of the Inverse Method of Fluxions, the Rationale of the Transmutation of the Origin of Fluents to other
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places at pleasure, the way of finding the contemporaneous Increments of Fluents, and such like.

Then to conclude the Method of Fluxions, a very convenient and general Method is proposed and explain'd, for the Resolution of all kinds of Equations, Algebraical or Fluxional, by having recourse to superior orders of Fluxions. This Method indeed is not contain'd in our Author's present Work, but is contrived in pursuance of a notable hint he gives us, in another part of his Writings. And this Method is exemplify'd by several curious and useful Problems.

Lastly, by way of Supplement or Appendix, some Terms in the Mathematical Language are farther explain'd, which frequently occur in the foregoing work, and which it is very necessary to apprehend rightly. And a sort of Analytical Praxis is adjoin'd to this Explanation, to make it the more plain and intelligible; in which is exhibited a more direct and methodical way of resolving such Algebraical or Geometrical Problems as are usually proposed; or an attempt is made, to teach us to argue more closely, distinctly, and Analytically.

And this is chiefly the substance of my Comment upon this part of our Author's work, in which my conduct has always been, to endeavour to digest and explain every thing in the most direct and natural order, and to derive it from the most immediate and genuine Principles. I have always put myself in the place of a Learner, and have endeavour'd to make such Explanations, or to form this into such an Institution of Fluxions and infinite Series, as I imagined would have been useful and acceptable to myself, at the time when I first enter'd upon these Speculations. Matters of a trite and easy nature I have pass'd over with a slight animadversion: But in things of more novelty, or greater difficulty, I have always thought myself obliged to be more copious and explicite; and am conscious to myself, that I have every where proceeded *cum sincero animo docendi*. Wherever I have fallen short of this design, it should not be imputed to any want of care or good intentions, but rather to the want of skill, or to the abstruse nature of the subject. I shall be glad to see my defects supply'd by abler hands, and shall always be willing and thankful to be better instructed.

What perhaps will give the greatest difficulty, and may furnish most matter of objection, as I apprehend, will be the Explanations before given, of *Moments, vanishing quantities, infinitely little quantities,*

ties, and the like, which our Author makes use of in this Treatise, and elsewhere, for deducing and demonstrating his Method of Fluxions. I shall therefore here add a word or two to my foregoing Explanations, in hopes farther to clear up this matter. And this seems to be the more necessary, because many difficulties have been already started about the abstract nature of these quantities, and by what name they ought to be call'd. It has even been pretended, that they are utterly impossible, inconceivable, and unintelligible, and it may therefore be thought to follow, that the Conclusions derived by their means must be precarious at least, if not erroneous and impossible.

Now to remove this difficulty it should be observed, that the only Symbol made use of by our Author to denote these quantities, is the letter *o*, either by itself, or affected by some Coefficient. But this Symbol *o* at first represents a finite and ordinary quantity, which must be understood to diminish continually, and as it were by local Motion; till after some certain time it is quite exhausted, and terminates in mere nothing. This is surely a very intelligible Notion. But to go on. In its approach towards nothing, and just before it becomes absolute nothing, or is quite exhausted, it must necessarily pass through vanishing quantities of all proportions. For it cannot pass from being an assignable quantity to nothing at once; that were to proceed *per saltum*, and not continually, which is contrary to the Supposition. While it is an assignable quantity, tho' ever so little, it is not yet the exact truth, in geometrical rigor, but only an Approximation to it; and to be accurately true, it must be less than any assignable quantity whatsoever, that is, it must be a vanishing quantity. Therefore the Conception of a Moment, or vanishing quantity, must be admitted as a rational Notion.

But it has been pretended, that the Mind cannot conceive quantity to be so far diminish'd, and such quantities as these are represented as impossible. Now I cannot perceive, even if this impossibility were granted, that the Argumentation would be at all affected by it, or that the Conclusions would be the less certain. The impossibility of Conception may arise from the narrowness and imperfection of our Faculties, and not from any inconsistency in the nature of the thing. So that we need not be very solicitous about the positive nature of these quantities, which are so volatile, subtle, and fugitive, as to escape our Imagination; nor need we be much in pain, by what name they are to be call'd; but we may confine ourselves wholly to the use of them, and to discover their properties.

properties. They are not introduced for their own sakes, but only as so many intermediate steps, to bring us to the knowledge of other quantities, which are real, intelligible, and required to be known. It is sufficient that we arrive at them by a regular progress of diminution, and by a just and necessary way of reasoning; and that they are afterwards duly eliminated, and leave us intelligible and indubitable Conclusions. For this will always be the consequence, let the *media* of ratiocination be what they will, when we argue according to the strict Rules of Art. And it is a very common thing in Geometry, to make impossible and absurd Suppositions, which is the same thing as to introduce impossible quantities, and by their means to discover truth.

We have an instance similar to this, in another species of Quantities, which, though as inconceivable and as impossible as these can be, yet when they arise in Computations, they do not affect the Conclusion with their impossibility, except when they ought so to do; but when they are duly eliminated, by just Methods of Reduction, the Conclusion always remains sound and good. These Quantities are those Quadratick Surds, which are distinguish'd by the name of impossible and imaginary Quantities; such as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, &c. For they import, that a quantity or number is to be found, which multiply'd by itself shall produce a negative quantity; which is manifestly impossible. And yet these quantities have all varieties of proportion to one another, as those foregoing are proportional to the possible and intelligible numbers 1, $\sqrt{2}$, $\sqrt{3}$, 2, &c. respectively; and when they arise in Computations, and are regularly eliminated and excluded, they always leave a just and good Conclusion.

Thus, for Example, if we had the Cubick Equation $x^3 - 12x^2 + 41x - 42 = 0$, from whence we were to extract the Root x ; by proceeding according to Rule, we should have this surd Expression for the Root, $x = 4 + \sqrt[3]{3 + \sqrt{-\frac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\frac{100}{27}}}$, in which the impossible quantity $\sqrt{-\frac{100}{27}}$ is involved; and yet this Expression ought not to be rejected as absurd and useless, because, by a due Reduction, we may derive the true Roots of the Equation from it. For when the Cubick Root of the first *vinculum* is rightly extracted, it will be found to be the impossible Number $-1 + \sqrt{-\frac{4}{3}}$, as may appear by cubing; and when the Cubick Root of the second *vinculum* is extracted, it will be found to be $-1 - \sqrt{-\frac{4}{3}}$. Then by collecting these Numbers, the

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impossible Number $\sqrt{-\frac{4}{3}}$ will be eliminated, and the Root of the Equation will be found $x = 4 - 1 - 1 = 2$.

Or the Cubick Root of the first *vinculum* will also be $\frac{3}{2} + \sqrt{-\frac{1}{12}}$, as may likewise appear by Involution; and of the second *vinculum* it will be $\frac{3}{2} - \sqrt{-\frac{1}{12}}$. So that another of the Roots of the given Equation will be $x = 4 + \frac{3}{2} + \frac{3}{2} = 7$. Or the Cubick Root of the same first *vinculum* will be $-\frac{1}{2} - \sqrt{-\frac{25}{12}}$; and of the second will be $-\frac{1}{2} + \sqrt{-\frac{25}{12}}$. So that the third Root of the given Equation will be $x = 4 - \frac{1}{2} - \frac{1}{2} = 3$. And in like manner in all other Cubick Equations, when the surd *vincula* include an impossible quantity, by extracting the Cubick Roots, and then by collecting, the impossible parts will be excluded, and the three Roots of the Equation will be found, which will always be possible. But when the aforesaid surd *vincula* do not include an impossible quantity, then by Extraction one possible Root only will be found, and an impossibility will affect the other two Roots, or will remain (as it ought) in the Conclusion. And a like judgment may be made of higher degrees of Equations.

So that these impossible quantities, in all these and many other instances that might be produced, are so far from infecting or destroying the truth of these Conclusions, that they are the necessary means and helps of discovering it. And why may we not conclude the same of that other species of impossible quantities, if they must needs be thought and call'd so? Surely it may be allow'd, that if these Moments and infinitely little Quantities are to be esteem'd a kind of impossible Quantities, yet nevertheless they may be made useful, they may assist us, by a just way of Argumentation, in finding the Relations of Velocities, or Fluxions, or other possible Quantities required. And finally, being themselves duly eliminated and excluded, they may leave us finite, possible, and intelligible Equations, or Relations of Quantities.

Therefore the admitting and retaining these Quantities, however impossible they may seem to be, the investigating their Properties with our utmost industry, and applying those Properties to use whenever occasion offers, is only keeping within the Rules of Reason and Analogy; and is also following the Example of our sagacious and illustrious Author, who of all others has the greatest right to be our Precedent in these matters. 'Tis enlarging the number of general Principles and Methods, which will always greatly

contribute to the Advancement of true Science. In short, it will enable us to make a much greater progress and proficiencie, than we otherwise can do, in cultivating and improving what I have elsewhere call'd *The Philosophy of Quantity*.

F I N I S.



Newton, Isaac,

The Method of Fluxions And Infinite Series With Its Application to the
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